FLOW OF POWER LAW FLUIDS
WITH APPLICATION TO OIL DRILLING

E.J. CARTER

A thesis submitted in partial fulfilment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

of the

COUNCIL FOR NATIONAL ACADEMIC AWARDS

LONDON

PLYMOUTH POLYTECHNIC

IN COLLABORATION WITH ENGLISH CHINA CLAYS LTD.,
ST. AUSTELL, CORNWALL

OCTOBER 1986

TO MY PARENTS

ACKNOWLEDGEMENTS

My sincere thanks are expressed to my supervisors, Dr. J.M. Davies and Dr. T.E.R. Jones for their help, encouragement and support during the course of the work.

My thanks also, to Dr. P. Lundie of English China Clays Ltd., St. Austell, for his advice and interest in the earlier part of the work.

My thanks are expressed also to the Science Research Council whose grant made available the services of the Interactive Computing Facility at the Rutherford Laboratory at Chiltern. My thanks to all those at the Rutherford Laboratory for their instruction and advice.

The Computer Services Unit at the Polytechnic are thanked for their support.

The Governors of the Polytechnic are thanked for their award of the research assistantship.

My thanks, too, to Angela and Debbie for all the typing they have done.

Lastly, my thanks to Margaret, Tania, Katharine and Jonathan.

FLOW OF POWER LAW FLUIDS

WITH APPLICATION TO OIL DRILLING

E.J. CARTER

ABSTRACT

The thesis is concerned with a theoretical study of the flow behaviour of inelastic power law fluids in two different types of flow situation. These are:

1. The creeping motion of a sphere moving through an expanse of liquid.

2. The combined steady and oscillatory flow of a liquid through a straight tube of circular cross section.

The first part of the work is devoted to the prediction of the drag correction factor for a sphere falling slowly through a bounded inelastic power law fluid. The analysis is carried out for the case when the outer spherical boundary has a finite or infinite radius. A perturbation technique is used to produce the resulting equations for a slightly power law fluid which are solved using the finite element method. An asymptotic expansion is used to provide an analytical far field solution for the infinite outer sphere case.

The second part considers the combined steady and oscillatory flow of an inelastic power law liquid in a tube. The analysis is carried out for the case when both the steady flow rate and the oscillatory flow rate are known. An expression for the pressure gradient reduction in the tube is then derived. The resulting partial differential equation is solved by finite difference techniques. An analytical solution for the pressure gradient is also obtained using a perturbation analysis for the case when the fluid inertial effects are small.

CONTENTS

CHAPTER 1

INTRODUCTION

Since ancient times the study of the deformation and flow of matter has been both a subject of speculation and of practical application. A discipline, drawing its inspiration from mathematics and engineering and epitomised by such a person as Archimedes (285-215 BC), who was one of the early workers in this field. The modern mathematical foundations were laid down in the 17th Century when Robert Hooke presented his law relating the stress and strain in an elastic solid and Isaac Newton formulated his law of friction for viscous fluids. At this time materials were classified either as Hookean solids or as Newtonian fluids. Newton, however, had considered only situations involving simple shear in the fluid and the generalisation of his work to include arbitrary types of flow was carried out by G.G. Stokes. In 1849 Stokes derived the now celebrated Navier-Stokes equations. Between the work of Hooke and Newton and that of Stokes and Navier lay all the development of the science of fluid mechanics created by such mathematicians as D'Alembert, Lagrange and Johannes and Daniel Bernouili by developing and extending the fundamental work of Euler. The subject was then further developed by such scientists as Helmholtz, Kelvin and Rayleigh during the 19th Century.

With the coming of the industrial revolution predicting and using the behaviour of fluid flows became very important. Understanding played a subservient role to application and empirical relationships were widely used. Equally important, new materials were being produced with properties very different from those previously considered. It was this development in material technology which finally demonstrated the inadequacy of the old way of classifying substances. Experimental work by Bingham and Green (1919) on oil paints and, later, German research on the flow of colloidal solutions, showed that a whole class of fluids existed whose behaviour could not be described by Newton's law of friction. The oil paints exhibited a yield stress, the colloidal solutions a shear rate dependent viscosity. In 1924 Ostwald considered the flow of a pseudoplastic fluid over a wide range of shear rates. In order to account for this behaviour Ostwald proposed the idea of a structural viscosity. This interest in the underlying structural foundations of flow phenomena has continued to the present day. Molecular theories have recently attracted a lot of interest and stand as alternatives to the more usual continuum mechanics approach (Bird[1] (1977)).

Materials were also examined which were found to possess both viscous and elastic behaviour. These materials were classified according to their response to small applied stresses. This classification is still used today. Materials not continually changing their shape when subjected to small applied stresses were classified as elastico-viscous solids. Materials which did change shape continually when subjected to a small applied stress (however small) were classified as visco-elastic fluids. These fluids have a 'memory' of past events and the ability to store energy when work is done on them. Elastic properties are exhibited by many different types of fluid including emulsions, suspensions and polymer solutions.

This growth of knowledge marked the beginning of a new science - rheology - and in 1929 the (American) Society of Rheology was founded.

In Chapter 2, in order to facilitate the understanding of the studies in this thesis, and to put them into their rheological perspective, some general aspects of the theory of deformation and flow of a homogeneous and continuous material are considered. In particular the important work of J.G. Oldroyd concerning the characterisation and formulation of rheological equations of state is discussed. The generalised Newtonian fluids and the more general Oldroyd four constant fluids, the simple fluid of Coleman and Noll and the Rivlin-Ericksen fluids are considered. In the second part of the chapter the applicability of the generalised Newtonian fluids in the oil drilling industry is discussed. It is established that, in the flow regime in which drill cuttings are transported to the surface, or when they settle after drilling is halted, the power law fluid accurately models many modern drilling fluids.

In order to introduce the numerical techniques used in Chapter 5 the Galerkin formulation for the finite element method is discussed in Chapter 3. The finite element method is a discretization technique which enables the approximate solution of a partial differential equation to be obtained at discrete points in its domain of definition. This is achieved by solving a set of simultaneous linear equations constrained by the relevant boundary conditions. The Galerkin formulation for the finite element method was chosen because of its proven success in fluid flow problems (Refer T.J. Chung[2] (1978)).

In Chapter 4 an analysis of the creeping flow of a power law fluid past a sphere is presented. The analysis is carried through for both a finite and an infinite expanse of fluid. This fluid is considered to be slightly power law (i.e. to have a power law index near 1). Its governing equation is derived using a perturbation analysis about $n=1$. An expression is derived for the drag force on the sphere, first in a finite, and then in an infinite, expanse of fluid. Using the latter expression for the drag force on the sphere the drag correction factor is shown to be a linear relation of the power law index. This linear relation is known to within a constant which is dependent on derivatives of the stream function of the power law fluid. Thus, in order to determine the drag correction factor the governing equation of the fluid must be solved, numerically, to obtain the stream function. The outer boundary condition required in the case of an infinite expanse of fluid was obtained by first deriving an expression for the asymptotic far field stream function and then using the field matching technique.

In Chapter 5, the numerical solution for the stream function was obtained by decoupling the governing equation into a system of two second order partial differential equations. These equations were then solved using the Galerkin formulation of the finite element method. In this chapter the full numerical procedure for obtaining the drag correction factor for a sphere falling slowly in a slightly power law fluid is presented. In addition the validation of the program and the numerical procedures used to obtain intermediate results are discussed. The work of other researchers in this area is commented upon and the result obtained in this chapter is compared with their predictions.

In Chapter 6, a theoretical study of the pulsatile flow of a power law fluid through a long, straight horizontal pipe of circular cross section is presented. Experimental work has shown that when a sinusoidal pressure gradient is superimposed onto the steady flow of visco-elastic fluid in a straight tube the mean flow rate is increased for a given mean pressure gradient. This phenomenon has been called 'flow enhancement' and is of considerable industrial interest. In the conventional pulsatile flow apparatus the mean pressure gradient is controlled and the mean flow rate measured. In our study the mean flow rate and the pulsatile flow rate are assumed known and the mean pressure gradient is evaluated. The relevant equation of motion is solved numerically by using a finite difference scheme. A perturbation analysis was also carried out for the case when fluid inertia is small and the results obtained are compared with the corresponding numerical results.

CHAPTER 2

RHEOLOGICAL EQUATIONS OF STATE

2.1 Introduction

This thesis is concerned with two flow situations:

(i) a sphere falling slowly through an infinite expanse of power law fluid.

(ii) the pumping of a power law fluid through a long horizontal pipe, of circular cross section, under the action of a pulsatile pressure gradient with a non-zero mean.

Both of these flow situations have industrial significance. The first flow situation is of particular importance to the oil drilling industry since many oil drilling fluids conform to the power law model and drill cuttings may be modelled, to a first approximation by spheres. The industrial significance of the second flow situation is self evident and is discussed in Chapter 6.

To facilitate the understanding of the studies in this thesis and to put them into their rheological perspective some general aspects of the theory of deformation and flow of a homogeneous and continuous material (fluid) are considered. First, some basic principles are stated and discussed and then some equations of state are considered. These equations of state are divided into two classes, those describing the generalised Newtonian fluids and those describing more general fluids - namely the Oldroyd, Simple and Rivlin-Ericksen fluids. The chapter concludes with a short discussion of the use of the generalised Newtonian fluids to the oil drilling industry.

2.2 Characterization and Formulation

Firstly, equations of state are obtained which specify the rheological properties of an arbitrary element of the fluid.

Secondly, the behaviour of the fluid in bulk, under the given initial and boundary conditions and the body forces operating, is predicted. This is achieved by solving, the equation of motion in conjunction with the equation of continuity.

In this thesis the component notation is used throughout. According to this approach a tensor is synonymous with its set of components, this set transforming covariantly, contravariantly or in mixed mode under a transformation of the coordinate system used. A tensor may accordingly be written as T_{ij}, T^{ij}, $T^{i}{}_{j}$ or $T_{i}{}^{j}$.

The simplest model of a fluid is the Newtonian fluid given by

$$e^{(1)}_{ii} = 0 \qquad \text{and} \qquad p_{ik} = -p\, g_{ik} + p'_{ik} \qquad (2.1)$$

where

$$p'_{ik} = 2\eta\, e^{(1)}_{ik} \qquad (2.2)$$

$e^{(1)}_{ik}$ = rate of strain tensor

p_{ik} = Cauchy stress tensor

p = an arbitrary isotropic pressure

g_{ik} = the metric tensor of a fixed coordinate system x^i

η = a scalar constant (the fluid viscosity)

p'_{ik} = extra stress tensor

The extra stress tensor is that part of the stress tensor associated with changes in shape of the fluid element. In terms of the velocity vector v_i, the rate of strain tensor is given by:

$$e^{(1)}_{ik} = \frac{1}{2}\left(v_{i,k} + v_{k,i} \right) \qquad (2.3)$$

The equation of motion and the equation of continuity are given respectively by:

$$p^{ki}{}_{,i} + \rho F^{i} = \rho \frac{Dv^{i}}{Dt} \qquad (2.4)$$

and

$$\frac{D\rho}{Dt} + \rho v^{i}{}_{,i} = 0 \qquad (2.5)$$

where ρ is the fluid density, F is the body force per unit mass, v is the fluid element velocity and D/Dt is the material derivative. For an incompressible fluid $D\rho/Dt = 0$.

A Newtonian fluid is quite inadequate for describing the complex rheological behaviour of many fluids. The most interesting fluids are the visco-elastic fluids. Under a shearing deformation these fluids behave neither as a perfectly elastic solid nor as a perfectly viscous fluid. The work done in a shearing deformation of such a visco-elastic fluid is neither totally dissipated as heat nor is it totally conserved. The conservation of work done manifests itself in many interesting phenomena including the die-swell effect, Weissenberg rod climbing and various strain-recovery effects. Visco-elastic fluid behaviour is still not fully understood, a fact that is witnessed by the numerous modern papers on the subject.

Very important developments took place in continuuum theoretical rheology during the 1950's and onwards. Progress in this country was largely due to the pioneering work of the late J.G. Oldroyd[3] and was inaugurated by him in his paper of 1950. This paper comprehensively discusses the general principles for the formation of equations of state and the application of the general theory through the construction of models.

The principles to be satisfied in constructing a model were stated by Oldroyd as follows:

(1) the equation must describe fluid properties independently of the frame of reference.

(2) the behaviour of any fluid element is to be dependent only upon its own deformation history.

Note (1) The usual convention of summing over repeated suffixes is complied with.

Note (2) A suffix following a comma indicates a covariant derivative

(3) the behaviour of any fluid element is to be independent of any translatory or rotational motion of the fluid as a whole.

The system of co-ordinates used by Oldroyd was that proposed by H. Hencky[4] in 1925, namely the system of convected co-ordinates. These co-ordinates are related to surfaces in the fluid which transform continuously with the fluid during its motion. They are independent of time and label the material points of the fluid. More significantly they allow the principles for constructing fluid models to be satisfied.

The visco-elastic behaviour of fluids can be modelled by mechanical analogy. Dashpots and springs may be taken as perfectly viscous and perfectly elastic elements respectively and various parallel and series combinations can be constructed. In many visco-elastic fluids memory effects are important and the superposition principle of L. Boltzmann[5] (1876) becomes applicable. Oldroyd[6] (1958) discusses a material which in simple shearing flow may be rheologically modelled by a viscous and an elastic element in series where the latter has the elastic after-effect properties consistent with Boltzmann's principle.

Oldroyd's work provides a constructive framework and guiding principles for the formulation of equations of state. Other workers, notably Rivlin and Ericksen[7] (1955), Green and Rivlin[8,9] (1957, 1960) and Coleman and Noll[10-12] (1960, 1961, 1961) have approached the problem of formulation in a different way from Oldroyd.

A significant simplification is achieved if only those motions are considered in which non-linear effects in the elastic and viscous deformation process may be ignored. These motions are those in which small deformations (solids) or small rates of strain (fluids) only occur. Such circumstances arise in experiments widely used by physical chemists to study the molecular structure of fluids and in rheological experiments when small amplitude oscillatory flows are considered.

The general linearized equations of state for an incompressible fluid are given by K. Walters[13] (1960) as

$$p_{ik} = -p\,g_{ik} + p'_{ik} \qquad (2.6)$$

where

$$p'_{ik} = 2\int_{-\infty}^{t} \Psi(t-t')\, e^{(1)}_{ik}(x,t')\,dt' \qquad (2.7)$$

$$\Psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau} \exp\left(\frac{t'-t}{\tau}\right) d\tau \qquad (2.8)$$

and

τ ≡ the relaxation time

$N(\tau)$ ≡ the relaxation spectrum over τ

A brief description of some equations of state now follows.

2.3 Equations of State

2.3.1 The Generalised Newtonian Fluids

The models discussed here are empirical and formulated by allowing the viscosity to be dependent either upon the rate of strain or upon the extra stress tensor.

The viscosity of a fluid is a scalar quantity and, if made dependent upon the rate of strain, it must be a function of the invariants of the rate of strain tensor $e^{(1)}_{ik}$. These invariants are I, II and III. For an incompressible fluid I = Div $\underline{v}$ = 0, while for a shearing flow III = 0 (see R.B. Bird[14] 1977, page 207). Thus the viscosity η may be written as:

$$\eta = \eta(\dot{\gamma})$$

where

$$2\dot{\gamma}^2 = \mathrm{II}$$

and

$\dot{\gamma}$ = shear rate

Thus the Newtonian fluid may be generalised to give

$$p'_{ik} = \eta(\dot{\gamma})\, e^{(1)}_{ik}$$

particular fluid models being formed by taking different functions $\eta(\dot{\gamma})$.

The models considered here are:

the Bingham fluid,
the power law fluid,
the Herschel-Bulkley fluid,
Robertson-Stiff fluid,
The Casson fluid.

The Ellis model is included as an example where the viscosity is considered to be a function of the extra stress.

The Bingham Fluid

In this model

$$\eta = \infty \qquad \tau \leq \tau_0 \qquad (2.12)$$

$$\eta = \mu_0 + \frac{\tau_0}{\dot{\gamma}} \qquad \tau > \tau_0 \qquad (2.13)$$

where

$\tau \equiv$ magnitude of p'_{ik}
$\dot{\gamma} \equiv$ shear rate

and τ_0, μ_0 are two parameters, the first one being identifiable as the yield stress of the fluid. Since, for a generalised Newtonian fluid,

$$p'_{ik} = \eta(\dot{\gamma})\, e^{(1)}_{ik} \qquad (2.14)$$

then

$$\tau = \eta(\dot{\gamma})\,\dot{\gamma}$$

and

$$\tau - \tau_0 = \mu_0 \dot{\gamma}$$

The Power Law Fluid

In this model $\eta(\dot{\gamma}) = K\dot{\gamma}^{n-1}$ where K and n are parameters. K is called the consistency factor and n the power law index.

But

$$\tau = \eta(\dot{\gamma})\,\dot{\gamma}$$

ie

$$\tau = K\dot{\gamma}^{n}$$

K is the viscosity when the shear rate is $1 s^{-1}$, effectively it indicates how viscous a fluid is at rest. When n=1 the model reduces to a Newtonian fluid.

The Herschel-Bulkley Fluid

This is a three parameter model given by:

$$\tau = \tau_o + K\dot{\gamma}^n \qquad (2.19)$$

It allows for a yield stress and a power law relationship between shear stress and shear rate when this yield stress is exceeded.

The Robertson-Stiff Fluid

This is also a three parameter model and is given by:

$$\tau = A(\dot{\gamma} + C)^B \qquad (2.20)$$

The Newtonian fluid (B=1, C=0), the Bingham fluid (B=1, C≠0) and the power law fluid (B≠1, C=0) are all special cases of this fluid.

The Casson Fluid

This model was first used in 1959 to describe the flow of pigment-oil suspensions in shearing flows. It is given by:

$$\eta^{1/2} = \eta_\infty^{1/2} + C\dot{\gamma}^{-1/2} \qquad (2.21)$$

and

$$\tau = \eta\dot{\gamma} \qquad (2.22)$$

When $\dot{\gamma}$ is very large $\eta \simeq \eta_\infty$ (the viscosity at infinite shear rate). The degree of shear thinning is given by C, the Casson slope.

The Ellis Fluid

In this case

$$\frac{\eta_o}{\eta} = 1 + \left(\frac{\tau}{\tau_{1/2}}\right)^{\alpha-1} \qquad (2.23)$$

where

$\eta_o \equiv$ zero shear-rate viscosity

$\tau_{\frac{1}{2}} \equiv$ shear stress at which $\eta = \frac{1}{2}\eta_o$

2.3.2 More General Fluids

The Oldroyd Fluids

This class of fluid arose from the general discussion presented by J.G. Oldroyd[6] in his 1958 paper. The 4 constant model equation of state is given by:

$$p'^{ik} + \lambda_1 \frac{\vartheta p}{\vartheta t}'^{ik} + \mu_0 p'_j{}^j e^{(1)ik} = 2\eta_0 e^{(1)ik} + 2\eta_0 \lambda_2 e^{(2)ik} \qquad (2.24)$$

where the material constants μ_0, η_0, λ_1 and λ_2 satisfy the inequalities

$$\eta_0 > 0 \qquad (2.25)$$

$$9\lambda_2 > \lambda_1 \geq \lambda_2 > 0 \qquad (2.26)$$

and $\vartheta/\vartheta t$ is a time derivative which does not imply any dependence on absolute motion in space.

This model produces realistic values for shear stress and first normal stress difference over a fairly wide range of shear rates. However, a zero second normal stress difference is predicted in simple shear flows.

The Simple Fluid of Coleman and Noll

This fluid was proposed by B. Coleman and W. Noll in 1961. The equation of state was given by:

$$p_{ik}(x,t) = p'_{ik} - p\,g_{ik} \qquad (2.27)$$

where

$$p'_{ik}(x,t) = \underset{-\infty}{\overset{t}{F}}\left[C_{ik}(x,t,t')\right] \qquad (2.28)$$

and F is a functional of the Cauchy deformation tensor C_{ik}. The equation of state has been given with respect to a fixed co-ordinate system.

Since a visco-elastic fluid necessarily has a fading memory this behaviour must be incorporated in the model through the functional F. Coleman and Noll specified an influence function h(t-t') and used it to define a norm, $||C_{ik}||$, of the strain history, where

$$\| C_{ik} \|^2 = \int_0^\infty C^2_{ik}(s)\, h^2(s)\, ds \qquad (2.29)$$

Here s = t-t', where t is the current time and t' an earlier time.

Finally they were able to approximate their functional by:

$$\int_{-\infty}^{t} M_1(t-t')\,C_{ik}(t')\,dt'$$

$$+\int_{-\infty}^{t}\int_{-\infty}^{t} M_2(t-t',\,t-t'')\,C_{ij}(t')\,C_{ij}(t'')\,dt'\,dt''$$

$$+\ \text{ORDER}\ \|C_{ik}\|^3 \qquad (2.30)$$

To a first order approximation the equation of state may be written as:

$$P_{ik} = P'_{ik} - p\,g_{ik} \qquad (2.31)$$

where

$$p'_{ik} = \int_{-\infty}^{t} M_1(t-t')\,C_{ik}(t')\,dt' \qquad (2.32)$$

These equations are called the equations of finite linear elasticity.

In 1966 A.S. Lodge[15] showed the simple fluids to be a special case of Oldroyd's visco-elastic fluids.

The Rivlin-Ericksen Fluids

These fluids may be considered to be approximations to the simple fluids. The models are valid for retarded motion or rapidly fading memory (K. Walters[16] 1970). The equations of state are constructed on the assumption that the stress is a function of the velocity gradients and the acceleration gradients up to the (n-1)th. The stress is deduced to be a function of the first n Rivlin-Ericksen tensors $A_{ik}^{(n)}$, where $A_{ik}^{(n)} = 2e_{ik}^{(n)}$.

The lower order Rivlin-Ericksen fluids can be shown to be equivalent to

$$p'_{ik} = 2\alpha_1 e^{(1)}_{ik} \qquad (2.33)$$

(First order)

$$p'_{ik} = 2\alpha_1 e^{(1)}_{ik} + 2\alpha_2 e^{(2)}_{ik} + 4\alpha_3 e^{(1)j}_{i}\, e^{(1)}_{jk}$$

(Second order)

and

$$
\begin{aligned}
p'_{ik} = {} & 2\alpha_1 e^{(1)}_{ik} + 2\alpha_2 e^{(2)}_{ik} + 4\alpha_3 e^{(1)}_i{}^j e^{(1)}_{jk} \\
& + 8\beta_1 e^{(1)}_j{}^\ell e^{(1)}_\ell{}^j e^{(1)}_{ik} + 2\beta_2 e^{(3)}_{ik} \\
& + 4\alpha_5 \left(e^{(1)}_i{}^j e^{(2)}_{jk} + e^{(2)}_i{}^j e^{(1)}_{jk} \right)
\end{aligned}
\qquad (2.35)
$$

(Third order)

where α_1, α_2, α_3, α_5 and β_1, β_2 are all material constants.

2.4 Some Industrial Applications of the Generalised Newtonian Fluids

2.4.1 Introduction to Oil Drilling Fluids

The oil drilling fluids used today are both rheologically complex and highly developed to perform many different tasks. A far from exhaustive list of these tasks would include the following:

Cleaning of the hole.
Cooling and lubricating the bit and drill string.
Preventing cave-in of the drilled formation.
Suspending drill cuttings when drilling has stopped (e.g. to change the bit).
Transporting cuttings to the surface during drilling.

With the highly deviated boreholes currently being drilled, rheological studies of oil drilling fluids are of more interest to the drilling industry than at any previous time. This was the stimulus that prompted the studies of Chapters 4 - 6 of this thesis.

The base material for a drilling fluid is either a clay/water dispersion, an oil/water emulsion or a polymer. A very wide range of additives has been developed and a drilling fluid is a mixture of its base fluid and many of these additives. The drilling situation determines which formulation is most suitable for the drilling fluid to be used and practical field experience is vital at this stage.

The most commonly used drilling fluids are the water/clay based ones since oil based drilling fluids are expensive and polymer based drilling fluids degrade, both biologically and under high temperatures. To the water/clay base barium sulphate (barite) and a suitable polymer are usually added. The barite is a weight material ensuring adequate pressure at the bit, the polymer is a viscosity modifier.

Although very varied, drilling fluids are, as a rule, non-Newtonian, pseudoplastic and thixotropic.

A detailed reference for the formulation, nature and function of oil drilling fluids is the book 'Drilling and Drilling Fluids' by G.V. Chilingarian and P. Vorabutr[17] (1981).

2.4.2 The Field Application of Model Fluids

A fluid model is required which is easily applied at the well-site. No model is valid over the whole of any of the flow regimes and engineering approximations are resorted to. From this it can be seen that the use of sophisticated models is unsuitable in the field and major efforts have been focused instead on the use of simple models allied to computer facilities at the well-site.

The Bingham fluid has long been used as the standard model but has recently been partly replaced by the power law model which still continues to gain favour. The Herschel-Bulkley and Robertson-Stiff fluids are also beginning to achieve proper acceptance in the field while in the last five years the Casson model has also been used.

Figure 1 (after F. Bagshaw, Company Report of Kelco Oil Field Products) shows the shear stress - shear rate relationship for one of their commercial drilling fluids. The instrument used to achieve this being a Fann viscometer. A Bingham fluid and a power law fluid were constructed from the data to represent the real drilling fluid. The shear stress - shear rate behaviour of these fluids is also given on the figure to allow comparison. The Kelco drilling fluid may be taken as fairly typical. The power law fluid provides a better representation than the Bingham fluid over the annular shear-rate range. This is particularly true over the lower end of that range. At shear rates above 300 s^{-1} the simpler Bingham fluid provides an equally good representation of the behaviour of the oil drilling fluid. In the field the parameters of the Bingham fluid are fixed by readings taken at just two shear rates, namely 400 s^{-1} and 900 s^{-1}.

The power law fluid has enjoyed wide usage in the field only in the last ten years or so. It usefully describes the upward flow regime of many modern drilling fluids. However, neither the Bingham fluid nor the power law fluid properly describes the flow situation at the bit where the very high shear rates are experienced (R. Lauzon, K. Reid[18] (1979)).

The more simple clay/water mixtures can approach plastic behaviour while many polymer based drilling fluids exhibit power law behaviour. However most drilling fluids exhibit both a yield stress and a non-linear shear stress - shear rate relationship when the yield stress has been exceeded (M. Zamora and R. Bleier[19] (1976)).

The Herschel-Bulkely and Robertson-Stiff fluids are suitable in this case and were used by the oil drilling industry for the first time in 1976. Robertston and Stiff[20] (1976) showed their model to be significantly better than the power law fluid, for the three different drilling fluids they considered, at describing the shear stress - shear rate behaviour over a shear rate range of $5s^{-1}$ to $1000s^{-1}$. However, obtaining the parameters for this model is not without its difficulties (Zamora, Bleier[19] (1976)).

A drilling fluid may best be represented, over a wide shear rate range, by several power-law regions. Such composite power-law behaviour can be described by the Casson fluid, used by the oil-drilling industry since 1979. Its main advantages (R. Lauzon, K. Reid[18] (1979)) are that it is valid from very low to very high shear rates and is easily implemented in the field.

D. McEachern[21] (1966) used the Ellis model to describe his experimental results for the laminar flow of dilute polymer solutions in an annulus and found it to be superior to the power-law fluid. The polymer solutions used were of a type relevant to the oil-drilling industry but as yet the Ellis fluid has been little, if at all, used in the field.

A further discussion of the types of rheological behaviour found in drilling fluids is given by M. Slawomirski[22] (1975).

CHAPTER 3

THE GALERKIN FORMULATION FOR THE FINITE ELEMENT METHOD

3.1 Introduction

In this chapter the Galerkin formulation of the finite element method is described in sufficient detail to explain its application in Chapter 5. The modern use of this method started in the 1940's when the techniques were applied to the structural design of aircraft. From these beginnings the method has developed to the point where today it is applied extensively by engineers to solve a wide range of problems.

In Chapter 5 the numerical procedures used to obtain the drag correction factor for a sphere falling slowly through an infinite expanse of power law fluid are presented. These procedures require the solution of the governing equation of a power law fluid. This is a fourth order partial diffential equation. The solution is obtained by decoupling it into a pair of second order partial differential equations and applying the finite element method within an iterative routine. Due to its past use in fluid dynamics (Chung (1978)) the Galerkin formulation of the method was the one implemented. This formulation, due to both Bubnov and Galerkin, is often called the method of weighted residuals.

3.2 The Galerkin Formulation

Let the equation to be solved approximately by the Galerkin method be

$$Au = f \qquad (3.1)$$

where A is a differential operator.
Equation (3.1) is defined over a physical region, Ω, with boundary Γ, and is solved subject to the relevant boundary conditions on Γ. The approximate solution, $\hat{u}$, is expressed as a linear combination of

a set of n linearly independent functions, ϕ_i, which satisfy the boundary conditions.

i.e.

$$\hat{u} = \sum_{i=1}^{n} c_i \phi_i \qquad (3.2)$$

where c_i : i = 1,...,n are constants.

The functions ϕ_i : i=1,...,n are called the basis, or shape, functions and $\hat{u}$ is called the nth trial function. The coefficients c_i must be such that, as n increases, the resultant sequence of trial functions is convergent to the true solution. The Galerkin formulation is a particular method for deriving the coefficients c_i.

Since $\hat{u}$ is an approximation to the true solution u

$$A\hat{u} - f = r \qquad (3.3)$$

where r is called the residual and equals the error incurred when $\hat{u}$, and not u, occurs in equation (3.1). An inner product $\langle r, w_i \rangle$ is now constructed where w_i : i=1,...,n is a set of weighting functions to be defined. The residual, r, is now forced to zero, in an average sense, over Ω by setting

$$\langle r, w_i \rangle = 0 \qquad i=1,\ldots,n \qquad (3.4)$$

The weighting functions may be chosen in different ways and this has led to four different methods: the method of moments, the method of least squares, the collocation method and the Galerkin method. In the Galerkin method we set

$$w_i = \phi_i \qquad (3.5)$$

and

$$\langle r, w_i \rangle = \int_{\Omega} r\, w_i \, d\Omega \qquad i=1,\ldots,n \qquad (3.6)$$

Thus

$$\int_{\Omega} r\, \phi_i \, d\Omega = 0 \qquad i=1,\ldots,n \qquad (3.7)$$

Equations (3.7) are solved simultaneously for c_i : i=1,...,n and these values are then substituted into equation (3.2) to give the approximate solution of the given differential equation.

In order to implement the work above, the basis functions ϕ_i : i=1,...,n must be known. The construction of these functions is described in the next section.

3.3 Constructing the Finite Element Mesh and the Local Basis Functions

Due to the symmetry of the flow situation considered in Chapter 5, our problem may be considered to be a two dimensional one from the outset. Consequently, the implementation of the finite element method is discussed only over plane regions Ω.

First the plane region is subdivided into an equivalent set of finite elements. The simplest example would be the subdivision of a rectangular area, by means of a grid, into a collection of squares or rectangles. These squares or rectangles form the finite elements and the grid forms the finite element mesh. If the plane area has a curved boundary then the collection of squares and rectangles will not coincide exactly with it. A better approximation will be produced if a mesh of quadilateral elements is used. In practice, many hundreds (or thousands) of elements may be used and a close approximation to the physical region of interest can be obtained.

Nodes are now allocated to each element in the mesh. A node is always allocated to each element vertex and may also be allocated to the element mid-side or its interior. Eight noded and four noded quadrilateral elements are often used.

It is necessary to construct the finite elements and the mesh because the basis functions $\phi_i : i=1,\dots,n$ are difficult to define as functions over the whole of the physical area. However, by means of a local coordinate system attached to each finite element, local basis functions may be easily defined. Using these local basis functions, equations equivalent to (3.7) but valid only over each of the finite elements separately can be obtained. When these equations have been obtained for every one of the finite elements, the results may be combined to obtain the required equation (3.7).

Some further aspects of this procedure are now discussed. Four noded plane quadrilateral elements have local basis functions defined over them which allow for a bilinear variation with respect to the local coordinate system. This local coordinate system is defined in Fig. 2(a). The nodes have been labelled locally, that is with respect to the element and not with respect to the grid. Numbering nodes with respect to the grid is called global node numbering. The local coordinates of the vertices of the finite element, and a typical interior point, are given in Fig. 2(b). In Fig. 2(c), it is shown that every finite element can be represented by a square with centre the origin, if the local coordinate system is referred to normal rectilinear axes with equal scales. The local shape functions are given by

$$\begin{aligned} \phi_1^{(e)} &= \tfrac{1}{4}(1-u)(1-v) \\ \phi_2^{(e)} &= \tfrac{1}{4}(1-u)(1+v) \\ \phi_3^{(e)} &= \tfrac{1}{4}(1+u)(1+v) \\ \phi_4^{(e)} &= \tfrac{1}{4}(1+u)(1-v) \end{aligned} \qquad (3.8)$$

for all points within the element and on its boundary, and zero at all exterior points. We have

$$\begin{aligned} \phi_i^{(e)} &= 1 \quad \text{at node } i \\ &= 0 \quad \text{at the other nodes} \end{aligned} \qquad (3.9)$$

Also

$$\hat{u}^{(e)} = \sum_{i=1}^{4} c_i^{(e)} \phi_i^{(e)} \qquad (3.10)$$

over the element (e). From equation (3.9) and (3.10) we have

$$c_i^{(e)} = \hat{u}_i^{(e)} \qquad (3.11)$$

where $\hat{u}_i^{(e)}$ is the approximate value of u at the ith node of the element (e). Thus

$$\hat{u}^{(e)} = \sum_{i=1}^{4} \hat{u}_i^{(e)} \phi_i^{(e)} \qquad (3.12)$$

In Chapter 5, the finite elements used are isoparametric. That is we assume equations (3.12) to be true for the global coordinates $(U^{(e)}, V^{(e)})$ of a point in element (e).

i.e.

$$U^{(e)} = \sum_{i=1}^{4} U_i^{(e)} \phi_i^{(e)} \qquad (3.13)$$

and

$$V^{(e)} = \sum_{i=1}^{4} V_i^{(e)} \phi_i^{(e)} \qquad (3.14)$$

where $(U_i^{(e)}, V_i^{(e)})$ are the global coordinates of the ith local node of element (e).

3.4 Constructing the Stiffness Equation

3.4.1 The General Method

Green's theorem in the plane is applied to equations (3.7) and the resulting equations are applied to each of the finite elements in turn. After using equation (3.12), implementing the necessary changes of variable and carrying out the numerical integration we obtain, for each element (e)

$$S^{(e)} \hat{\underline{u}}^{(e)} = \underline{\ell}^{(e)} \qquad (3.15)$$

where

$$\hat{\underline{u}}^{(e)} = \left[u_i^{(e)} : i = 1,2,3,4 \right] \qquad (3.16)$$

$$\underline{\ell}^{(e)} = \left[\ell_i^{(e)} : i = 1,2,3,4 \right] \qquad (3.17)$$

$S^{(e)}$ is the element stiffness matrix, $\ell^{(e)}$ the element load vector. The element stiffness equations are used to construct the full stiffness equation given by

$$S \hat{u} = \underline{\ell} \qquad (3.18)$$

where

$$\hat{\underline{u}} = \left[\hat{u}_i : i = 1,\dots,N \right] \qquad (3.19)$$

N = total number of nodes in mesh

and

$\hat{u}_i$ is the approximate value of u at the ith globally numbered node of the mesh.

3.4.2 An Example

The following example shows how equation (3.18) is derived from equations (3.15). We consider the very simple two element mesh given in Fig. 2(d). In this figure the global labels of the nodes are shown on the outside of the mesh, the local labels are shown on the inside. The element stiffness matrices are $S^{(1)}$ and $S^{(2)}$ and the full stiffness matrix is S. The matrix $S^{(e)}$ contributes $S^{(e)}_{i,j}$, to $S_{G_i^{(e)},G_j^{(e)}}$ where $G_i^{(e)}$ and $G_j^{(e)}$ are the global node numbers corresponding to local node numbers i and j in element (e). The elements in S are obtained by adding together the contributions made by $S^{(1)}$ and $S^{(2)}$ separately. For example

$$S_{2,3} = S^{(1)}_{1,3} + S^{(2)}_{2,1} \qquad (3.20)$$

If neither element has a contribution to make then that element in S is zero. For example

$$S_{4,1} = 0 \qquad (3.21)$$

Two important points to note are

(i) If the element stiffness matrices are symmetric then it follows that the full stiffness matrix is also symmetric.

(ii) The full stiffness matrix will be banded (i.e. its non-zero elements will be grouped about the leading diagonal). The width of the band depends upon the way in which the global numbering of the nodes is carried out.

The stiffness matrices used in the numerical work presented in Chapter 5 were, in some cases, of such a high order that techniques were developed to modify them so that only the non-zero elements were considered.

Details of the boundary conditions and their implementation are given in Chapter 5.

CHAPTER 4

THEORY OF THE MOTION OF A SPHERE IN A POWER LAW FLUID

4.1 Introduction

The ability of a drilling fluid to convey cuttings from a well is not fully understood and this is particularly so in highly deviated wells where difficulties are frequently experienced in cleaning the hole. The cuttings travel with a lower velocity than the drilling fluids and these cuttings can accumulate in the well bore. If this is not kept to a minimum then it can lead either to the degradation of the cuttings or the drill string may get lodged in the hole. Experimental work by P. Reynolds[23] (1982) has shown that, in a power law fluid, the settling velocity of an irregular shaped particle may be adequately represented by the settling velocity of a sphere having the same volume and density. In the case of the fluid model, it has been well established by the oil drilling industry that, in the shear rate range of interest, many modern drilling fluids exhibit power law behaviour. Therefore, in this thesis we have studied the problem of the flow of a power law fluid past a stationary sphere. The effect of fluid inertia is ignored and the fluid is considered to be slightly power law. That is its power law index is near unity.

The theoretical studies of this problem have, in the past, been based mainly on perturbation and variational methods (Acharya et al[24] (1976)). Wasserman and Slattery[25] (1964) used variational principles applied to inelastic power law fluids to obtain upper and lower bounds of the drag correction factor as the power law index varied from 1.0 to 0.2. These bounds have been improved by Mohan and Venkateshwarlu[26] (1976), by Yoshioka and Adachi[27] (1973) and by Cho and Hartnett[28] (1983). Chhabra[29] (1983) has commented on the results of Cho and Hartnett and has provided a comparison of their results with experimental values.

More recently Crochet et al[30] (1984) and Gu Dazhi and Tanner[31] (1985) have used a finite element analysis. Crochet et al considered a sphere in a cylinder geometry with the ratio of the radius of the cylinder to the radius of the sphere being equal to 50. Gu Dazhi and Tanner used a sphere within a sphere configuration and an extrapolation technique to obtain the drag force on the falling sphere when the fluid is infinite in extent. They note that wall effects are important for a slightly power law fluid but for a power law index less than, or equal to 0.5 they are negligible. The analysis in this chapter, although valid only when the power law index is near unity, uses a field matching technique to solve the equations when the fluid is unbounded. The use of this technique was influenced by a paper by Tiefenbruck and Leal[32] (1979). Other workers who have used discretization techniques include Bush and Phan-Thien[33] (1985). In their case the drag force on a sphere in creeping motion through a Carreau fluid was calculated.

4.2 The Governing Equations for a Slightly Power Law Fluid

We consider the slow flow of a power law fluid between two concentric spheres. The inner sphere is fixed and the outer sphere is given an instantaneous velocity U in the positive z direction. A spherical polar coordinate system (r, θ, ϕ) is used with its origin at the centre of the fixed sphere. Since the flow situation is axi-symmetrical a velocity distribution of the form

$$V_r = V_r(r, \theta), \quad V_\theta = V_\theta(r, \theta), \quad V_\phi = 0 \qquad (4\cdot1)$$

may be assumed. We consider first the case when the outer sphere has a finite radius and later extend this work to the case of an outer sphere with an 'infinite' radius. The schematic diagram of the flow situation used when the outer sphere has a finite radius is given in Fig. 3.

In spherical polar coordinates the equation of continuity for axi-symmetrical flow is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(V_\theta \sin\theta\right) = 0 \qquad (4\cdot2)$$

We may therefore define a stream function ψ by the equations

$$V_r = \frac{-1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \tag{4.3}$$

$$V_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} \tag{4.4}$$

where V_r and V_θ satisfy the equation of continuity. When inertial terms are neglected the equations of motion become

$$-\frac{\partial p}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \tau_{rr}\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{\theta r} \sin\theta\right) - \frac{1}{r}\left(\tau_{\theta\theta} + \tau_{\phi\phi}\right) \tag{4.5}$$

and

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 \tau_{r\theta}\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{\theta\theta} \sin\theta\right) - \frac{\tau_{\phi\phi}}{r} \cot\theta \tag{4.6}$$

where τ_{ik} is the stress tensor. The equation of state is

$$\tau_{ik} = -\eta(\dot{\gamma})\dot{\gamma}_{ik} \tag{4.7}$$

where $\dot{\gamma}_{ik}$ is the rate of strain tensor with magnitude $\dot{\gamma}$ and viscosity

$$\eta(\dot{\gamma}) = K\dot{\gamma}^{n-1} \tag{4.8}$$

where K is a constant and n is the power law index. When n = 1 the equation of state refers to a Newtonian fluid and

$$\tau_{ik} = -K\dot{\gamma}_{ik} \tag{4.9}$$

The governing equation for a Newtonian fluid is derived from equations (4.2) - (4.6) and (4.9). Full details of the derivation are given in Appendix 1. The resulting equation is

$$E^4 \psi = 0 \tag{4.10}$$

where

$$E^4 \psi = E^2 (E^2 \psi) \tag{4.11}$$

and

$$E^2 \psi \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right] \tag{4.12}$$

The governing equation for a power law fluid is derived from equations (4.2) - (4.7) and (4.10). Equation (4.7) can be rewritten as

$$\tau_{ik} = -K\left(\frac{II}{2}\right)^{\frac{n-1}{2}} \dot{\gamma}_{ik} \qquad (4.13)$$

where II is the second invariant of $\dot{\gamma}_{ik}$ and is given by

$$II = \dot{\gamma}_{rr}^2 + \dot{\gamma}_{\theta\theta}^2 + \dot{\gamma}_{\phi\phi}^2 + 2\dot{\gamma}_{r\theta}^2 \qquad (4.14)$$

Substituting the components of the rate of strain tensor from equation (4.13) into the equations of motion (4.5) and (4.6)

$$\frac{\partial p}{\partial r} = K\left(\frac{II}{2}\right)^{\frac{n-1}{2}}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\dot{\gamma}_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\dot{\gamma}_{r\theta}\right) - \frac{1}{r}\left(\dot{\gamma}_{\theta\theta} - \dot{\gamma}_{\phi\phi}\right) + \left(\frac{n-1}{2}\right)\left(\frac{\dot{\gamma}_{rr}}{II}\frac{\partial II}{\partial r} + \frac{\dot{\gamma}_{r\theta}}{rII}\frac{\partial II}{\partial\theta}\right)\right] \qquad (4.15)$$

and

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = K\left(\frac{II}{2}\right)^{\frac{n-1}{2}}\left[\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\dot{\gamma}_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\dot{\gamma}_{\theta\theta}\sin\theta\right) - \frac{\dot{\gamma}_{\phi\phi}}{r}\cot\theta + \left(\frac{n-1}{2}\right)\left(\frac{\dot{\gamma}_{r\theta}}{II}\frac{\partial II}{\partial r} + \frac{\dot{\gamma}_{\theta\theta}}{rII}\frac{\partial II}{\partial\theta}\right)\right] \qquad (4.16)$$

Since a slightly power law fluid is considered then we can ignore terms of order $(n-1)^2$ and higher in our analysis.
Using

$$\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial p}{\partial\theta}\right) \qquad (4.17)$$

from equations (4.15), (4.16) and (4.17) we obtain

$$\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\dot{\gamma}_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\dot{\gamma}_{r\theta}\right) - \frac{1}{r}\left(\dot{\gamma}_{\theta\theta} + \dot{\gamma}_{\phi\phi}\right)\right]$$

$$+\left(\frac{n-1}{2}\right)\frac{\partial}{\partial\theta}\left[\frac{\dot{\gamma}_{rr}}{II}\frac{\partial II}{\partial r} + \frac{\dot{\gamma}_{r\theta}}{rII}\frac{\partial II}{\partial\theta}\right]$$

$$+\left(\frac{n-1}{2}\right)\frac{\partial II}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\dot{\gamma}_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\dot{\gamma}_{r\theta}\right) - \left(\frac{\dot{\gamma}_{\theta\theta} + \dot{\gamma}_{\phi\phi}}{r}\right)\right]$$

$$= \frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\dot{\gamma}_{r\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\dot{\gamma}_{\theta\theta}\sin\theta\right) - \dot{\gamma}_{\phi\phi}\cot\theta\right]$$

$$+ \frac{n-1}{2}\frac{\partial}{\partial r}\left[r\frac{\dot{\gamma}_{r\theta}}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial r} + \frac{\dot{\gamma}_{\theta\theta}}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial\theta}\right]$$

$$+ \frac{n-1}{2}\frac{1}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\dot{\gamma}_{r\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\dot{\gamma}_{\theta\theta}\sin\theta\right) - \dot{\gamma}_{\phi\phi}\cot\theta\right] \qquad (4.18)$$

Which may be written in the form

$$\frac{\partial}{\partial r}A_1 - \frac{\partial}{\partial\theta}B_1 = \frac{n-1}{2}\left[\frac{\partial}{\partial\theta}C_1 - \frac{\partial}{\partial r}D_1 - \frac{1}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial r}A_1 + \frac{1}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial\theta}B_1\right] \qquad (4.19)$$

where

$$A_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\dot{\gamma}_{r\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\dot{\gamma}_{\theta\theta}\sin\theta\right) - \dot{\gamma}_{\phi\phi}\cot\theta \qquad (4.20)$$

$$B_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\dot{\gamma}_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\dot{\gamma}_{r\theta}\sin\theta\right) - \frac{\dot{\gamma}_{\theta\theta} + \dot{\gamma}_{\phi\phi}}{r} \qquad (4.21)$$

$$C_1 = \frac{\dot{\gamma}_{rr}}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial r} + \frac{\dot{\gamma}_{r\theta}}{r\,\mathrm{II}}\frac{\partial \mathrm{II}}{\partial\theta} \qquad (4.22)$$

and

$$D_1 = \frac{r\,\dot{\gamma}_{r\theta}}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial r} + \frac{\dot{\gamma}_{\theta\theta}}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial\theta} \qquad (4.23)$$

Following closely the analysis for a Newtonian fluid (see Appendix 1) equation (4.19) may be written as

$$E^4\psi = \left(\frac{n-1}{2}\right)F(\psi)\sin\theta \qquad (4.24)$$

where

$$F(\psi) = \frac{\partial}{\partial\theta}C_1 - \frac{\partial}{\partial r}D_1 - \frac{1}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial r}A_1 + \frac{1}{\mathrm{II}}\frac{\partial \mathrm{II}}{\partial\theta}B_1 \qquad (4.25)$$

Since n is near unity we may use a perturbation analysis to solve equation (4.24). The stream function, therefore, is given by

$$\psi = \psi_N + (n-1)\,\psi_{NN} \qquad (4.26)$$

where

ψ_N is the Newtonian stream function

and ψ_{NN} is the non-Newtonian stream function contribution

Substituting equation (4.26) into equation (4.24) and equating powers of (n-1) we have, on ignoring terms of order $(n-1)^2$ and higher and using equation (4.10)

$$E^4\psi_{NN} = \tfrac{1}{2}F(\psi_N)\sin\theta \qquad (4.27)$$

The boundary conditions for ψ_{NN} are similar to those for the Newtonian case (as shown in Appendix 2) and are given by

$$\psi_{NN}\big|_{r=a} = \psi_{NN}\big|_{r=b} = 0 \qquad (4.28)$$

$$\frac{\partial\psi_{NN}}{\partial r}\Big|_{r=a} = \frac{\partial\psi_{NN}}{\partial r}\Big|_{r=b} = 0 \qquad (4.29)$$

$$\frac{\partial\psi_{NN}}{\partial\theta}\Big|_{\theta=0} = \frac{\partial\psi_{NN}}{\partial\theta}\Big|_{\theta=\pi} = 0 \qquad (4.30)$$

In the infinite outer sphere case the boundary conditions for ψ_{NN} are discussed in the asymptotic far field section.

We shall now derive $F(\psi_N)$ which is required in equation (4.27). The following expressions are for a Newtonian velocity field

$$\dot{\gamma}_{rr} = 2\frac{\partial V_r}{\partial r} \qquad (4.31)$$

$$\dot{\gamma}_{\theta\theta} = 2\left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r}\right) \qquad (4.32)$$

$$\dot{\gamma}_{\phi\phi} = 2\left(\frac{V_r}{r} + V_\theta\frac{\cot\theta}{r}\right) \qquad (4.33)$$

and

$$\dot{\gamma}_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r} + \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right) \qquad (4.34)$$

From Appendix 3 equation (A3.11)

$$\psi_N = -\frac{1}{2}Ua^2\left(A\frac{a}{r} + B\frac{r}{a} + C\frac{r^2}{a^2} + D\frac{r^4}{a^4}\right)\sin^2\theta \qquad (4.35)$$

and ψ_N is given in terms of V_r and V_θ by equations (4.3) and (4.4). Substituting these expressions into equations (4.20) and (4.21) we have

$$A_1 = -\frac{U}{a}\left(10D\frac{r}{a} + B\frac{a^2}{r^2}\right)\sin\theta \qquad (4.36)$$

$$B_1 = \frac{2U}{r^2}\left(5D\frac{r^2}{a^2} - B\frac{a}{r}\right)\cos\theta \qquad (4.37)$$

We now need to define the following

$$G_1 = \frac{2U}{a}\left(-3A\frac{a^4}{r^4} - B\frac{a^2}{r^2} + 2D\frac{r}{a}\right)\cos\theta \qquad (4.38)$$

$$G_2 = \dot{\gamma}_{\theta\theta} = -\frac{1}{2}G_1 \qquad (4.39)$$

$$G_3 = \dot{\gamma}_{\phi\phi} = G_2 \qquad (4.40)$$

$$G_4 = \dot{\gamma}_{r\theta} = \frac{3U}{a}\left(A\frac{a^4}{r^4} + D\frac{r}{a}\right)\sin\theta \qquad (4.41)$$

$$P_1 = \mathrm{II} = \frac{3}{2}G_1^2 + 2G_4^2 \qquad (4.42)$$

$$G_5 = \frac{\partial\dot{\gamma}_{rr}}{\partial r} = \frac{2U}{a^2}\left(12A\frac{a^5}{r^5} + 2B\frac{a^2}{r^2} + 2D\right)\cos\theta \qquad (4.43)$$

$$G_6 = \frac{\partial\dot{\gamma}_{rr}}{\partial\theta} = \frac{2U}{a}\left(-3A\frac{a^4}{r^4} - B\frac{a^2}{r^2} + 2D\frac{r}{a}\right)\sin\theta \qquad (4.44)$$

$$G_7 = \frac{\partial\dot{\gamma}_{r\theta}}{\partial r} = -3\frac{U}{a^2}\left(-4A\frac{a^5}{r^5} + D\right)\sin\theta \qquad (4.45)$$

$$G_8 = \frac{\partial\dot{\gamma}_{r\theta}}{\partial\theta} = -3\frac{U}{a}\left(A\frac{a^4}{r^4} + D\frac{r}{a}\right)\cos\theta \qquad (4.46)$$

$$S_1 = \frac{\partial^2\dot{\gamma}_{rr}}{\partial r^2} = 2\frac{U}{a^3}\left(-60A\frac{a^6}{r^6} - 6B\frac{a^4}{r^4}\right)\cos\theta \qquad (4.47)$$

$$S_2 = \frac{\partial^2 \dot{\gamma}_{rr}}{\partial r \partial \theta} = -\frac{2U}{a^2}\left(12A\frac{a^5}{r^5} + 2B\frac{a^3}{r^3} + 2D\right)\sin\theta \tag{4.48}$$

$$S_3 = \frac{\partial^2 \dot{\gamma}_{rr}}{\partial \theta^2} = -G_1 \tag{4.49}$$

$$S_4 = \frac{\partial^2 \dot{\gamma}_{r\theta}}{\partial r^2} = -\frac{3U}{a^3}\left(20A\frac{a^6}{r^6}\right)\sin\theta \tag{4.50}$$

$$S_5 = \frac{\partial^2 \dot{\gamma}_{r\theta}}{\partial r \partial \theta} = -\frac{3U}{a^2}\left(-4A\frac{a^5}{r^5} + D\right)\cos\theta \tag{4.51}$$

$$S_6 = \frac{\partial^2 \dot{\gamma}_{r\theta}}{\partial \theta^2} = -G_4 \tag{4.52}$$

$$P_2 = \frac{\partial \mathrm{II}}{\partial r} = 3G_1G_5 + 4G_4G_7 \tag{4.53}$$

Let

$$P_3 = \frac{\partial \mathrm{II}}{\partial \theta} = 3G_1G_6 + 4G_4G_8 \tag{4.54}$$

$$P_4 = \frac{\partial^2 \mathrm{II}}{\partial r^2} = 3G_1S_1 + 3G_5^2 + 4G_4S_4 + 4G_7^2 \tag{4.55}$$

$$P_5 = \frac{\partial^2 \mathrm{II}}{\partial r \partial \theta} = 3G_1S_2 + 3G_6G_5 + 4G_4S_5 + 4G_8G_7 \tag{4.56}$$

Let

$$P_6 = \frac{\partial^2 \mathrm{II}}{\partial \theta^2} = 3G_1S_3 + 3G_6^2 + 4G_4S_6 + 4G_8^2 \tag{4.57}$$

$$C_1^* = \frac{\partial C_1}{\partial \theta}, \qquad D_1^* = \frac{\partial D_1}{\partial r} \tag{4.58}$$

Finally we have

$$F(\psi_N) = C_1^* - D_1^* - \frac{P_2A_1}{P_1} + \frac{P_3B_1}{P_1} \tag{4.59}$$

Having obtained an analytical expression for $F(\psi_N)$ we may now solve equation (4.27) and obtain the non-Newtonian perturbation stream function, ψ_{NN}, subject to the boundary conditions of equations (4.28)-(4.30)

4.3 Derivation of the Drag Force on a Sphere Moving Slowly Through a Slightly Power Law Fluid

4.3.1 Finite Outer Sphere Drag Force Calculation

The orientation of the components of the total stress tensor at the surface of the sphere are given in Fig. 4. The total stress tensor, π_{ik}, is given by

$$\pi_{ik} = p\,\delta_{ik} + \tau_{ik} \tag{4.60}$$

where τ_{ik} is the extra stress tensor

δ_{ik} is the unit tensor

The drag force, F_D, acting on the inner sphere is given by

$$F_D = 2\pi a^2 \int_0^{\pi} \left(\pi_{r\theta} \sin^2\theta - \pi_{rr} \sin\theta \cos\theta\right)\Big|_{r=a} d\theta \tag{4.61}$$

Integrating by parts we have

$$F_D = 2\pi a^2 \int_0^{\pi} \left(\pi_{r\theta} + \frac{1}{2}\frac{\partial \pi_{rr}}{\partial \theta}\right)\Big|_{r=a} \sin^2\theta \, d\theta \tag{4.62}$$

From equations (4.3), (4.4) and the no slip condition on r=a we obtain

$$\frac{\partial \psi}{\partial \theta}\Big|_{r=a} = \frac{\partial \psi}{\partial r}\Big|_{r=a} = 0 \tag{4.63}$$

Substituting equation (4.3) into equation (4.31) gives

$$\dot{\gamma}_{rr}\Big|_{r=a} = \frac{-2}{\sin\theta}\left[-\frac{2}{r^3}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial r}\right)\right]_{r=0} = 0 \qquad (4.64)$$

Therefore

$$\Pi_{rr}\Big|_{r=a} = p \qquad (4.65)$$

The drag force, F_D, on the inner sphere is now given by

$$F_D = 2\pi a^2 \int_0^{\pi} \left(\tau_{r\theta} + \frac{1}{2}\frac{\partial p}{\partial\theta}\right)\Big|_{r=a} \sin^2\theta \, d\theta \qquad (4.66)$$

Similarly, from equations (4.32) and (4.33)

$$\dot{\gamma}_{\theta\theta}\Big|_{r=a} = \dot{\gamma}_{\phi\phi}\Big|_{r=a} = 0 \qquad (4.67)$$

Therefore, on substituting equations (4.64) and (4.67) into equation (4.14), we have

$$\mathrm{II}\Big|_{r=a} = 2\dot{\gamma}_{r\theta}^{\,2} \qquad (4.68)$$

From equations (4.3), (4.4) and (4.34), using the no slip conditions on r=a we obtain

$$\dot{\gamma}_{r\theta}\Big|_{r=a} = \frac{1}{a\sin\theta}\frac{\partial^2\psi}{\partial r^2}\Big|_{r=a} \qquad (4.69)$$

From equation (4.6) we have

$$-\frac{1}{r}\frac{\partial p}{\partial\theta}\Big|_{r=a} = \frac{1}{r^3}\frac{\partial}{\partial r}\left(\tau_{r\theta}\, r^3\right)\Big|_{r=a} \qquad (4.70)$$

Substituting equations (4.13) and (4.69) into equation (4.70) we obtain

$$\frac{\partial p}{\partial\theta}\Big|_{r=a} = \left(\frac{\mathrm{II}}{2}\right)^{n-1}\Bigg|_{r=a} \frac{\cdot k}{\sin\theta}\left[\frac{\partial^3\psi}{\partial r^3} + (n-1)\left(\frac{\partial^3\psi}{\partial r^3} - \frac{3}{a}\frac{\partial^2\psi}{\partial r^2}\right)\right]_{r=a} \qquad (4.71)$$

From equations (4.13), (4.66) and (4.68) - (4.71) we have

$$F_D = 2\pi K a^2 \int_0^{\pi} \left[\frac{1}{a^2 \sin^2\theta} \left(\frac{\partial^2 \psi}{\partial r^2} \right)^2 \right]_{r=a}^{\frac{n-1}{2}} \left[\frac{1}{2} \frac{\partial^3 \psi}{\partial r^3} - \frac{1}{a} \frac{\partial^2 \psi}{\partial r^2} \right]$$

$$+ \left(\frac{n-1}{2} \right) \left[\frac{\partial^3 \psi}{\partial r^3} - \frac{3}{a} \frac{\partial^2 \psi}{\partial r^2} \right]_{r=a} \sin\theta \, d\theta \qquad (4.72)$$

We now define the non-dimensional variables α_2, α_3, β_2 and β_3 by the equations

$$U\alpha_2 = \left. \frac{\partial^2 \psi_N}{\partial r^2} \right|_{r=a} \qquad (4.73)$$

$$\frac{U}{a} \alpha_3 = \left. \frac{\partial^3 \psi_N}{\partial r^3} \right|_{r=a} \qquad (4.74)$$

$$U\beta_2 = \left. \frac{\partial^2 \psi_{NN}}{\partial r^2} \right|_{r=a} \qquad (4.75)$$

$$\frac{U}{a} \beta_3 = \left. \frac{\partial^3 \psi_{NN}}{\partial r^3} \right|_{r=a} \qquad (4.76)$$

In equation (4.72) we require the evaluation of the term

$$\left[\left(\frac{\partial^2 \psi}{\partial r^2} \right)^2 \right]_{r=a}^{\frac{n-1}{2}} = \left| \frac{\partial^2 \psi}{\partial r^2} \right|_{r=a}^{n-1} \qquad (4.77)$$

From equations (4.73), (4.75) and (4.26)

$$\left| \frac{\partial^2 \psi}{\partial r^2} \right|_{r=a}^{n-1} = U^{n-1} \left| \alpha_2 + (n-1)\beta_2 \right|^{n-1} \qquad (4.78)$$

Using Taylor's expansion

$$\left| \frac{\partial^2 \psi}{\partial r^2} \right|_{r=a}^{n-1} = U^{n-1} \left(|\alpha_2|^{n-1} + (n-1)^2 \beta_2 \alpha_2 |\alpha_2|^{n-3} \cdots \right) \qquad (4.79)$$

By ignoring terms of order $(n-1)^2$ and higher we obtain

$$\left. \frac{\partial^2 \psi}{\partial r^2} \right|_{r=a}^{n-1} = U^{n-1} |\alpha_2|^{n-1} \qquad (4.80)$$

Using equation (4.35), equation (4.73) becomes

$$U\alpha_2 = \left.\frac{\partial^2 \Psi_N}{\partial r^2}\right|_{r=a} = -\frac{1}{2} U a^2 \frac{\partial^2}{\partial r^2}\left[A\frac{a}{r} + B\frac{r}{a} + C\frac{r^2}{a^2} + D\frac{r^4}{a^4}\right]_{r=a} \sin^2\theta \qquad (4.81)$$

Therefore

$$\alpha_2 = -\left[A + C + 6D\right]\sin^2\theta \qquad (4.82)$$

The coefficients A, C and D are expressed in terms of d, the ratio of the diameter of the outer sphere to the inner sphere. These expressions are derived in Appendix 3 and are given in equations (A3.31), (A3.29) and (A3.28). Using these equations we obtain

$$\alpha_2 = \frac{-3d(2d^4 + 2d^3 + 2d^2 - 3d - 3)\sin^2\theta}{(d-1)^3(4 + 7d + 4d^2)} \qquad (4.83)$$

By definition $d > 1$ and α_2 is negative. Thus equation (4.80) becomes

$$\left|\left.\frac{\partial^2 \psi}{\partial r^2}\right|_{r=a}\right|^{n-1} = U^{n-1}(-\alpha_2)^{n-1} \qquad (4.84)$$

Equation (4.72) may, therefore, be rewritten as

$$F_D = \pi K a^2 \left(\frac{U}{a}\right)^n \left[\int_0^\pi (-\alpha_2)^{n-1}\left(\alpha_3 - 2\alpha_2 + (n-1)(\alpha_3 - 3\alpha_2)\right)\sin^{1-n}\theta \sin\theta\, d\theta + (n-1)\int_0^\pi (-\alpha_2)^{n-1}(\beta_3 - 2\beta_2)\sin^{1-n}\theta \sin\theta\, d\theta\right] \qquad (4.85)$$

Since, for a slightly power law fluid, n-1 is near zero we may use the approximations

$$(-\alpha_2)^{n-1} = 1 + (n-1)\log(-\alpha_2) \qquad (4.86)$$

and

$$(\sin\theta)^{1-n} = 1 + (1-n)\log(\sin\theta) \qquad (4.87)$$

It is convenient to let

$$\alpha_2 = \alpha_2^* \sin^2\theta \tag{4.88}$$

$$\alpha_3 = \alpha_3^* \sin^2\theta \tag{4.89}$$

Substituting equations (4.86) - (4.89) into equation (4.85) we obtain

$$\begin{aligned} F_D = \pi K a^2 \left(\frac{U}{a}\right)^n \int_0^\pi \Big[& (\alpha_3^* - 2\alpha_2^*)\sin^3\theta \\ & + (n-1)(\alpha_3^* - 3\alpha_2^*)\sin^3\theta \\ & + (n-1)(\alpha_3^* - 2\alpha_2^*)(\log(-\alpha_2^*) + \log(\sin\theta))\sin^3\theta\Big]\, d\theta \\ & + \pi K a^2 \left(\frac{U}{a}\right)^n (n-1)\int_0^\pi (\beta_3 - 2\beta_2)\sin\theta\, d\theta \end{aligned} \tag{4.90}$$

From equations (4.35), (4.74) and (4.89) we obtain

$$\frac{U}{a}\alpha_3^* = -\frac{1}{2} U a^2 \frac{\partial^3}{\partial r^3}\left[A\frac{a}{r} + B\frac{r}{a} + C\frac{r^2}{a^2} + D\frac{r^4}{a^4}\right]_{r=a} \tag{4.91}$$

Therefore

$$\alpha_3^* = 3A - 12D \tag{4.92}$$

From equations (4.82) and (4.88)

$$\alpha_2^* = -A - C - 6D \tag{4.93}$$

In equation (4.90) we require the evaluation of the integrals I_1 and I_2 given by

$$I_1 = \int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$$

and (4.94)

$$I_2 = \int_0^\pi \sin^3\theta \log(\sin^2\theta)\, d\theta$$

From Appendix 4, equation (A4.13)

$$I_2 = -\frac{20}{9} + \frac{8}{3}\log 2 \qquad (4.95)$$

Substituting equations (4.92) - (4.95) into equation (4.90) we obtain

$$\begin{aligned} F_D = K a^2 \left(\frac{U}{a}\right)^n \Big[& \frac{4\pi}{3}(5A+2C) \\ & + 2\pi(n-1)\Big[(5A+2C)\left(\frac{2}{3}\log(2A+2C+12D) - \frac{5}{9}\right) \\ & + 4A + 2C + 4D\Big] \\ & + \pi(n-1)\int_0^\pi (\beta_3 - 2\beta_2)\sin\theta \, d\theta \Big] \end{aligned} \qquad (4.96)$$

where β_2 and β_3 are defined by equations (4.75) and (4.76) and are evaluated numerically by the method described in Chapter 5.

It should be noted that when the fluid is Newtonian (i.e. when n=1) the coefficients A and C are given by equations (A3.31) and (A3.29) and the drag force on the sphere becomes

$$F_D = 24\pi U \eta_0 a \frac{d(1+d+d^2+d^3+d^4)}{(d-1)^3(4+7d+4d^2)} \qquad (4.97)$$

where $K = \eta_0$ is the Newtonian viscosity.

4.3.2 The Drag Force for the Case of an Infinite Outer Sphere

The drag force for the infinite outer sphere problem can be obtained from the finite outer sphere case by letting d=b/a tend to infinity. In this case the constants A, C and D have been derived in Appendix 3 and are given by

$$A = 1/2 \qquad (4.98)$$
$$C = 1 \qquad (4.99)$$
$$D = 0 \qquad (4.100)$$

By substitution into equation (4.96) the drag force becomes

$$F_D = 6\pi a^2 k \left(\frac{U}{a}\right)^n \left[1 + (n-1)\left(\log 3 + \frac{1}{2}\right) + \frac{n-1}{6}\int_0^{\pi} (\beta_3 - 2\beta_2)\sin\theta \, d\theta\right] \qquad (4.101)$$

which may be written as

$$F_D = 6\pi a^2 k \left(\frac{U}{a}\right)^n \left[1 + (n-1)\alpha\right] \qquad (4.102)$$

where

$$\alpha = \log 3 + \frac{1}{2} + \frac{1}{6}\int_0^{\pi} (\beta_3 - 2\beta_2)\sin\theta \, d\theta \qquad (4.103)$$

However the drag force for a general power law fluid is usually expressed in the form

$$F_D = 12\pi a^2 k \left(\frac{U}{2a}\right)^n F(n) \qquad (4.104)$$

where F(n) is a function of n only and independent of the geometrical constants. On comparison with our expression for F_D in equation (4.102) we have

$$\frac{F(n)}{2^{n-1}} = 1 + (n-1)\alpha \qquad (4.105)$$

i.e.

$$F(n) = \left[1 + (n-1)\alpha\right] 2^{n-1} \qquad (4.106)$$

Ignoring terms of order $(n-1)^2$ and higher we have

$$2^{n-1} = 1 - (n-1)\log 2 \qquad (4.107)$$

and

$$F(n) = \left[1 + (n-1)(\alpha - \log 2)\right] \qquad (4.108)$$

Finally we have

$$F(n) = 1 + (n-1)\beta \qquad (4.109)$$

where, on using equation (4.106), we have

$$\beta = \log\frac{3}{2} + \frac{1}{2} + \frac{1}{6}\int_0^{\pi}(\beta_3 - 2\beta_2)\sin\theta\, d\theta \qquad (4.110)$$

where it should be noted that β_2 and β_3 are as defined in equations (4.75) and (4.76) and are functions of θ only.

β is a constant which is related to β_2 and β_3 and can therefore be obtained by solving equation (4.27) numerically and using equations (4.75) and (4.76). The outer boundary condition for equation (4.27) was found using a matching technique over a suitable imaginary sphere placed at a distance of ten or twenty times the radius of the inner sphere. However, to implement this technique we require the far field solution for the stream function.

4.4 The Asymptotic Far Field Stream Function for an Infinite Outer Sphere

The governing equation for a slightly power law fluid is given by equations (4.27) and (4.59). The asymptotic far field region is valid when r is large. In this region the stream function is given by

$$\psi_N = -\frac{Ua^2}{2}\left[\frac{1}{2}\frac{a}{r} - \frac{3}{2}\frac{r}{a} + \frac{r^2}{a^2}\right]\sin^2\theta \qquad (4.111)$$

From equations (4.36), (4.37) and (4.58) we obtain

$$A_1 = \frac{3Ua}{2r^2}\sin\theta \qquad (4.112)$$

$$B_1 = \frac{3Ua}{r^3}\cos\theta \qquad (4.113)$$

$$C_1^* = \frac{12Ua}{r^3}\sin\theta \qquad (4.114)$$

$$D_1^* = -\frac{6Ua}{r^3}\sin\theta \qquad (4.115)$$

and ignoring terms of order $1/r^m$ where $m > 4$ then equation (4.42) reduces to

$$\Pi = \frac{27U^2a^2}{2r^4}\cos^2\theta \qquad (4.116)$$

By substituting these expressions into equation (4.59) and using equation (4.27)

$$E^4\psi_{NN} = \frac{9Ua}{r^3}\sin^2\theta \qquad (4.117)$$

This equation was solved analytically by using a separable solution of the form

$$\psi_{NN} = g(r)\sin^2\theta \qquad (4.118)$$

Substituting into equation (4.117) we obtain

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)g(r) = \frac{9Ua}{r^3} \qquad (4.119)$$

Let

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)g(r) = g^*(r) \qquad (4.120)$$

By inspection, a particular integral of equation (4.119) is given by

$$g^*(r) = -\frac{3Ua\log r}{r} \qquad (4.121)$$

and equation (4.121) may be written as

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)g(r) = -\frac{3Ua\log r}{r} \qquad (4.122)$$

By inspection a particular integral of equation (4.122) is given by

$$g(r) = m r \log r + n r \qquad (4.123)$$

Comparing terms in equation (4.122) we obtain

$$\psi_p = \frac{3Ua}{2}\left(r \log r + \frac{r}{2}\right)\sin^2\theta \qquad (4.124)$$

as a particular integral of equation (4.119). The complementary function, which satisfies $E^4\psi = 0$ has the form

$$\psi_c = \left(\frac{A'}{r} + B'r + C'r^2 + D'r^4\right)\sin^2\theta \qquad (4.125)$$

where A', B', C' and D' are constants to be determined since

$$\psi_{NN} = \psi_c + \psi_p \qquad (4.126)$$

Then

$$\psi_{NN} = \left(\frac{A'}{r} + B'r + C'r^2 + D'r^4 + \frac{3Uar}{4} + \frac{3Uar\log r}{2}\right)\sin^2\theta \qquad (4.127)$$

From equations (4.111) and (4.127) the asymptotic far field value for ψ is given by

$$\psi = -\frac{Ua^2}{2}\left(\frac{a}{2r} - \frac{3r}{2a} + \frac{r^2}{a^2}\right)\sin^2\theta$$

$$+ (n-1)\left[\frac{A'}{r} + B'r + C'r^2 + D'r^4 + \frac{3Uar}{4} + \frac{3Uar\log r}{2}\right]\sin^2\theta \qquad (4.128)$$

As $r \to \infty$ we require that $\psi \to -\frac{Ur^2}{2}\sin^2\theta$, equation (4.128) may be written as

$$\psi = -\frac{1}{2}Ur^2\sin^2\theta . \left[1 + \frac{a^3}{2r^3} - \frac{3a}{2r} - (n-1)\frac{1}{U}\left(\frac{A'}{r^3} + \frac{B'}{r} + C' + D'r^2 + \frac{3Ua}{2r} + \frac{3Ua\log r}{r}\right)\right] \qquad (4.129)$$

For ψ to tend to the required limit

$$C' = D' = 0 \qquad (4.130)$$

and ψ_{NN} is given by

$$\psi_{NN} = \frac{3}{4}(n-1)\left(\frac{A^*}{r} + B^* r + Uar + 2Uar\log r\right)\sin^2\theta \quad (4.131)$$

where $$A^* = \frac{4}{3}A'$$

and $$B^* = \frac{4}{3}B' \quad (4.132)$$

The asymptotic far field is known to within the value of two unknown constants A* and B*.

The paper by Tiefenbruck and Leal[32] (1982) has shown that, when the outer sphere has an infinite radius, reasonable results for the drag force on the inner sphere can be obtained by matching the asymptotic far field into an imaginary spherical surface concentric with the inner sphere. The radius of this imaginary sphere must be greater than 10 times that of the inner sphere. The drag force over its surface is found using equation (4.96). The term A*/r was assumed small compared with B*/r for the matching radius used and was ignored. This assumption was justified by the fact that when the term was removed from the numerical routines the calculated values for the velocity field were unaffected.

The detailed method for obtaining the drag correction factor numerically, and the results achieved, are given in the next chapter.

CHAPTER 5

THE NUMERICAL SOLUTION FOR THE MOTION OF A SPHERE FALLING THROUGH AN INFINITE POWER LAW FLUID

5.1 Introduction

This chapter describes the numerical method that is used to solve the fourth order partial differential equation (4.27) which is presented in Chapter 4. This equation will be solved for both the finite outer sphere case and the infinite outer sphere case.

For the numerical solution we rewrite the fourth order partial differential equation in the following way

$$E^2 \psi_{NN} = \omega_{NN} \tag{5.1}$$

and

$$E^2 \omega_{NN} = \frac{1}{2} F(\psi_N) \sin\theta \tag{5.2}$$

where $F(\psi_N)$ is given by equation (4.59). It should be noted that ω_{NN} does not represent the non-Newtonian vorticity vector.

5.2 Boundary Conditions

We shall now require the boundary conditions for ψ_{NN}. Some of the boundary conditions for ψ_{NN} have already been presented in Chapter 4 but, for completeness, we shall also present the full boundary conditions.

5.2.1 Finite Outer Sphere Case

From equation (5.1) we have

$$\omega_{NN} = \frac{\partial^2 \psi_{NN}}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial \psi_{NN}}{\partial \theta} \right) \tag{5.3}$$

From equation (4.28) we have seen that $\psi_{NN}|_{r=a} = 0$. Therefore the first and second derivatives of ψ_{NN} with respect to θ evaluated on r=a are zero. Hence the boundary condition for ω_{NN} on r=a is given by

$$\omega_{NN}\Big|_{r=a} = \frac{\partial^2 \psi_{NN}}{\partial r^2}\Big|_{r=a} \tag{5.4}$$

Using a central finite difference scheme, we may approximate $\omega_{NN}|_{r=a}$ by the formula

$$\omega_{NN}\Big|_{r=a} = \frac{\psi_{NN}(a+dr,\theta)-2\psi_{NN}(a,\theta)+\psi_{NN}(a-dr,\theta)}{(dr)^2} \tag{5.5}$$

However, since $\partial\psi_{NN}/\partial r|_{r=a} = 0$ then

$$\psi_{NN}(a+dr,\theta) \simeq \psi(a-dr,\theta) \tag{5.6}$$

Therefore, on using the result $\psi_{NN}|_{r=a} = 0$ equation (5.5) simplifies to

$$\omega_{NN}\Big|_{r=a} = \frac{2\psi_{NN}(a+dr,\theta)}{(dr)^2} \tag{5.7}$$

Similarly, the boundary condition for ω_{NN} on r=b is given by

$$\omega_{NN}\Big|_{r=b} = 2\psi_{NN}(b-dr,\theta)/(dr)^2 \tag{5.8}$$

To determine the boundary condition for ω_{NN} on θ =0 we have to express ω_{NN} in a different form. Using equations (4.3) and (4.4) equation (5.3) may be rewritten in the form

$$\omega_{NN} = \frac{\partial^2 \psi_{NN}}{\partial r^2} + \sin\theta \frac{\partial}{\partial\theta}\left(V_{r_{NN}}\right) \tag{5.9}$$

where $V_{r_{NN}}$ is the non-Newtonian radial velocity component. By symmetry about θ =0 we have

$$\partial V_{r_{NN}} / \partial \theta = 0 \tag{5.10}$$

From equation (4.28) we also have $\psi_{NN}|_{r=a} = 0$ which implies that the first and second derivatives of ψ_{NN} with respect to r on θ =0 are both zero. Therefore, we have

$$\omega_{NN} \Big|_{\theta=0} = 0 \tag{5.11}$$

In a similar way, we may show that

$$\omega_{NN} \Big|_{\theta=\pi} = 0 \tag{5.12}$$

The complete boundary conditions for solving the partial differential equations (5.1) and (5.2) are therefore given by

$$\begin{aligned} \psi_{NN}\Big|_{r=a} &= \psi_{NN}\Big|_{r=b} = 0 \\ \psi_{NN}\Big|_{\theta=0} &= \psi_{NN}\Big|_{\theta=\pi} = 0 \\ \omega_{NN}\Big|_{\theta=0} &= \omega_{NN}\Big|_{\theta=\pi} = 0 \\ \omega_{NN}\Big|_{r=a} &= \frac{2\psi_{NN}(a+dr,\theta)}{(dr)^2} \end{aligned} \tag{5.13}$$

and

$$\omega_{NN}\Big|_{r=b} = \frac{2\psi_{NN}(b-dr,\theta)}{(dr)^2}$$

5.2.2 Infinite Outer Sphere Case

In this case the boundary conditions for ψ_{NN} and ω_{NN} are similar to those in equations (5.13). However, the boundary conditions on r=b are no longer required since ψ_{NN} and ω_{NN} are known for large r to within a constant B*. The new boundary conditions for ψ_{NN} and ω_{NN} are discussed in Section 5.3 and are given by

$$\psi_{NN}\Big|_{r=R} = \frac{3}{4}\left[B^*R + R + 2R\log R\right]\sin^2\theta \tag{5.14}$$

and

$$\omega_{NN}\Big|_{r=R} = -\frac{3}{2R}\left[B^* + 2\log R\right]\sin^2\theta \qquad (5.15)$$

where R is the radius of the matching spherical surface and B* is a constant which has to be determined by the iterative technique discussed in Section 5.3.

5.3 The Numerical Method

5.3.1 The Galerkin Formulation

From equation (4.12) equations (5.1) and (5.2) may be written as

$$\frac{\partial^2 \psi_{NN}}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi_{NN}}{\partial\theta}\right) = \omega_{NN} \qquad (5.16)$$

and

$$\frac{\partial^2 \omega_{NN}}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial \omega_{NN}}{\partial\theta}\right) = \frac{F(\psi_N)}{2}\sin\theta \qquad (5.17)$$

These equations have to be solved subject to the boundary conditions given in equations (5.13) - (5.15). The domain in the (r,θ) plane over which the finite element mesh is to be constructed is defined by $0 \leqslant \theta \leqslant \pi$ and $a \leqslant r \leqslant R$, where a is the radius of the inner sphere and R is the radius of the imaginary spherical matching surface (Fig. 5).

In order to remove the apparent numerical singularity on the boundaries $\theta = 0$ and $\theta = \pi$ equations (5.16) and (5.17) were transformed using the change of variables

$$r = e^z \qquad (5.18)$$

$$x = -\cos\theta \qquad (5.19)$$

The transformed equations are

$$\frac{\partial^2 \psi_{NN}}{\partial z^2} - \frac{\partial \psi_{NN}}{\partial z} + (1-x^2)\frac{\partial^2 \psi_{NN}}{\partial x^2} = e^{2z}\omega_{NN} \qquad (5.20)$$

and

$$\frac{\partial^2 \omega_{NN}}{\partial z^2} - \frac{\partial \omega_{NN}}{\partial z} + (1-x^2)\frac{\partial^2 \omega_{NN}}{\partial x^2} = e^{2z} f(e^z, x) \qquad (5.21)$$

The finite element mesh is bounded by $Z = \log a$, $Z = \log R$, $X = -1$ and $X = 1$. A regular finite element mesh was constructed over the (Z,X) plane (refer Program 1). From equation (5.18) it follows that in the (r,θ) plane the mesh has elements which, in the r direction, increase in size away from the inner sphere where both ψ_{NN} and ω_{NN} have their least variation. A typical mesh is shown in Fig. 6.

Let the finite element mesh constructed have N nodes and let $\{\phi_i : i = 1,\ldots,N\}$ be a set of global shape functions (refer Section 3.2). Then

$$\frac{\partial}{\partial Z}\left(\frac{\partial \psi_{NN}}{\partial Z}\phi_i\right) = \frac{\partial^2 \psi_{NN}}{\partial Z^2}\phi_i + \frac{\partial \psi_{NN}}{\partial Z}\frac{\partial \phi_i}{\partial Z} \qquad (5.22)$$

and

$$\frac{\partial}{\partial X}\left[(1-X^2)\frac{\partial \psi_{NN}}{\partial X}\phi_i\right] = (1-X^2)\frac{\partial^2 \psi_{NN}}{\partial X^2}\phi_i$$
$$+ (1-X^2)\frac{\partial \psi_{NN}}{\partial X}\frac{\partial \phi_i}{\partial X} - 2X\frac{\partial \psi_{NN}}{\partial X}\phi_i \qquad (5.23)$$

By Green's theorem in the plane

$$\int_{\Omega}\frac{\partial}{\partial Z}\left(\frac{\partial \psi_{NN}}{\partial Z}\phi_i\right)d\Omega = \int_{\Gamma}\phi_i\frac{\partial \psi_{NN}}{\partial Z}n_Z\,d\Gamma \qquad (5.24)$$

and

$$\int_{\Omega}\frac{\partial}{\partial X}\left((1-X^2)\frac{\partial \psi_{NN}}{\partial X}\phi_i\right)d\Omega = \int_{\Gamma}\phi_i(1-X^2)\frac{\partial \psi_{NN}}{\partial X}n_X\,d\Gamma \qquad (5.25)$$

where Ω is the area of the finite element mesh, Γ is its boundary and $(\partial\psi_{NN}/\partial Z)n_Z$ and $(\partial\psi_{NN}/\partial X)n_X$ are normal derivatives of ψ_{NN} on Γ (i.e. $\partial\psi_{NN}/\partial n$ on Γ).

The boundary conditions for $\partial\psi_{NN}/\partial n$ (and $\partial\omega_{NN}/\partial n$) on Γ are automatically satisfied when the boundary conditions for ψ_{NN} (and ω_{NN}) are satisfied. The test for this condition is given by Strang and Fix[34] (1973). Following A.J. Davies[35] (1980) the functions ϕ_i were chosen to be zero at all nodes on the boundary Γ.

Therefore

$$\int_{\Omega} \frac{\partial}{\partial z}\left(\frac{\partial \psi_{NN}}{\partial z} \phi_i\right) d\Omega = 0 \tag{5.26}$$

and

$$\int_{\Omega} \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial \psi_{NN}}{\partial x}\right] d\Omega = 0 \tag{5.27}$$

On multiplying equation (5.20) through by ϕ_i, integrating over Ω and using equations (5.26) and (5.27) we obtain

$$\int_{\Omega}\left[-\frac{\partial \psi_{NN}}{\partial z}\frac{\partial \phi_i}{\partial z} - (1-x^2)\frac{\partial \psi_{NN}}{\partial x}\frac{\partial \phi_i}{\partial x} + 2x\frac{\partial \psi_{NN}}{\partial x}\phi_i - \frac{\partial \psi_{NN}}{\partial z}\phi_i\right] d\Omega = \int_{\Omega} e^{2z}\omega_{NN}\phi_i \, d\Omega \tag{5.28}$$

for i=1,2,...,N.

Similarly, we obtain

$$\int_{\Omega}\left[-\frac{\partial \omega_{NN}}{\partial z}\frac{\partial \phi_i}{\partial z} - (1-x^2)\frac{\partial \omega_{NN}}{\partial x}\frac{\partial \phi_i}{\partial x} + 2x\frac{\partial \omega_{NN}}{\partial x}\phi_i - \frac{\partial \omega_{NN}}{\partial z}\phi_i\right] d\Omega = \int_{\Omega} e^{2z} f(e^z, x)\phi_i \, d\Omega \tag{5.29}$$

for i=1,...,N

Equations (5.28) and (5.29) were applied over each element in turn. The global shape functions were replaced by the local shape functions $\{\phi_j^e : j=1,2,3,4\}$ and the approximations

$$\psi_{NN} = \psi_{NN}^j \phi_j^e \qquad j = 1,2,3,4 \tag{5.30}$$

and

$$\omega_{NN} = \omega_{NN}^j \phi_j^e \qquad j = 1,2,3,4 \tag{5.31}$$

were substituted into equations (5.28) and (5.29). For boundary elements the local shape functions associated with nodes on Γ were set to zero. The integrals over the elements were approximated by the four point Gaussian quadrature rule. The resulting local stiffness matrices were combined, as described in Section 3.4, to form the global stiffness matrix, S. The element load vectors were similarly combined to form the global load vectors $L\psi_{NN}$ and $L\omega_{NN}$. The matrix equations obtained were

$$S\underline{\psi}_{NN} = L\psi_{NN} \tag{5.32}$$

and

$$S\underline{\omega}_{NN} = L\omega_{NN} \tag{5.33}$$

where $\underline{\psi}_{NN}$ is the column vector $[\psi^{1}_{NN}, \ldots, \psi^{N}_{NN}]$

and $\underline{\omega}_{NN}$ is the column vector $[\omega^{1}_{NN}, \ldots, \omega^{N}_{NN}]$

Here ψ^{i}_{NN} and ω^{i}_{NN} are the values of ψ_{NN} and ω_{NN}, respectively, at the ith node of the mesh.

In order to show how the boundary conditions ψ_{NN} are implemented numerically, equation (5.32) is written as

$$\begin{bmatrix} S_{11} & \cdots & \cdots & S_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ S_{i1} & \cdots & S_{ii} \cdots & S_{iN} \\ \cdots & \cdots & \cdots & \cdots \\ S_{N1} & \cdots & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} \psi^{1}_{NN} \\ \cdot \\ \psi^{i}_{NN} \\ \cdot \\ \psi^{N}_{NN} \end{bmatrix} = \begin{bmatrix} L^{1}\psi_{NN} \\ \cdot \\ L^{i}\psi_{NN} \\ \cdot \\ L^{N}\psi_{NN} \end{bmatrix} \tag{5.34}$$

Let the ith node of the mesh lie on Γ and let

$$\psi^{i}_{NN} = V \tag{5.35}$$

where the value of V is given by equation (5.13) or (5.14). Equation (5.34) was modified in the following way

$$\begin{bmatrix} S_{11} & \cdots & \cdots & S_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ S_{i1} & \cdots & S_{ii} \cdots & S_{iN} \\ \cdots & \cdots & \cdots & \cdots \\ S_{N1} & \cdots & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} \psi^{1}_{NN} \\ \cdot \\ \psi^{i}_{NN} \\ \cdot \\ \psi^{N}_{NN} \end{bmatrix} = \begin{bmatrix} L^{1}\psi_{NN} \\ \cdot \\ KS_{ii} \\ \cdot \\ L^{N}\psi_{NN} \end{bmatrix} \tag{5.36}$$

where K is chosen to be 10^{24} (a suitably large number). Therefore

$$S_{i1}\psi^{1}_{NN} + \cdots + K S_{ii}\psi^{i}_{NN} + \cdots + S_{iN}\psi^{N}_{NN} = K S_{ii} V \quad (5.37)$$

Since the terms containing K are dominant then equation (5.37) becomes

$$K S_{ii}\psi^{i}_{NN} \simeq K S_{ii} V \quad (5.38)$$

But $S_{ii} \neq 0$ and hence

$$\psi^{i}_{NN} = V \quad (5.39)$$

Therefore when equations (5.36) are solved simultaneously, the solution is constrained by the boundary conditions as required. The boundary conditions for ω_{NN} were implemented in exactly the same way as those for ψ_{NN}.

5.3.2 The Matching Technique

We have already seen from equation (4.131) that ψ_{NN} for large r is known to within a constant B*, when the term in A* is ignored, and is given by

$$\psi_{NN} = \frac{3}{4}\left(B^{*}r + r + 2r\log r\right)\sin^{2}\theta \quad (5.40)$$

Substituting ψ_{NN} from equation (5.40) into equation (5.3) then ω_{NN} is also known for large r in terms of the same constant B* and is given by

$$\omega_{NN} = -\frac{3}{2r}\left(B^{*} + 2\log r\right)\sin^{2}\theta \quad (5.41)$$

The iterative technique that is used to determine the unknown constant B* is now discussed.

The value of B* can be obtained in one of two ways.

(i) We consider $\partial\psi_{NN}/\partial r$ to be continuous across some imaginary spherical surface r=R i.e. $$\left.\frac{\partial \psi_{NN}}{\partial r}\right|_{r=R-} = \left.\frac{\partial \psi_{NN}}{\partial r}\right|_{r=R+} \qquad (5.42)$$

(ii) We consider $\partial \omega_{NN}/\partial r$ to be continuous across the imaginary boundary r=R

i.e. $$\left.\frac{\partial \omega_{NN}}{\partial r}\right|_{r=R-} = \left.\frac{\partial \omega_{NN}}{\partial r}\right|_{r=R+} \qquad (5.43)$$

The value of B* can be obtained either by matching $\partial \psi_{NN}/\partial r$ or by matching $\partial \omega_{NN}/\partial r$ across the imaginary spherical boundary r=R. Obviously we can only satisfy the derivatives of the stream function for a particular value of θ. In our case, this was chosen to be $\theta = \pi/2$. We shall now discuss these two methods in detail.

(i) We consider

$$\left.\frac{\partial \psi_{NN}}{\partial r}\right|_{\substack{r=R- \\ \theta=\frac{\pi}{2}}} = \left.\frac{\partial \psi_{NN}}{\partial r}\right|_{\substack{r=R+ \\ \theta=\frac{\pi}{2}}} \qquad (5.44)$$

Now $\psi_{NN}|_{r=R+}$ is given by equation (5.40). Hence, on differentiating equation (5.40) with respect to r and substituting into equation (5.42) we obtain

$$\left.\frac{\partial \psi_{NN}}{\partial r}\right|_{\substack{r=R- \\ \theta=\pi/2}} = \frac{3}{4}\left(B^{*} + 3 + 2\log r\right) \qquad (5.45)$$

By rearranging this equation we may express B* as

$$B^{*} = \frac{4}{3}\left.\frac{\partial \psi_{NN}}{\partial r}\right|_{\substack{r=R- \\ \theta=\pi/2}} - 3 - 2\log r \qquad (5.46)$$

It should be noted that $\partial \psi_{NN}/\partial r|_{r=R-}$ must be determined from the numerical stream function using a forward finite difference approximation.

(ii) We consider $\partial \omega_{NN}/\partial r$ to be continuous across the matching surface r=R.

i.e. $$\frac{\partial \omega_{NN}}{\partial r}\bigg|_{\substack{r=R- \\ \theta=\pi/2}} = \frac{\partial \omega_{NN}}{\partial r}\bigg|_{\substack{r=R+ \\ \theta=\pi/2}} \qquad (5.47)$$

We can determine $\partial \omega_{NN}/\partial r|_{\substack{r=R+ \\ \theta=\pi/2}}$ by differentiating equation (5.41). On substituting into equation (5.43) we obtain

$$\frac{\partial \omega_{NN}}{\partial r}\bigg|_{\substack{r=R- \\ \theta=\pi/2}} = \frac{3}{2r^2}\left(B^* - 2 + 2\log r\right) \qquad (5.48)$$

On rearranging this equation we may express B* as

$$B^* = \frac{2}{3} r^2 \frac{\partial \omega_{NN}}{\partial r}\bigg|_{\substack{r=R- \\ \theta=\pi/2}} + 2 - 2\log r \qquad (5.49)$$

We again note that $\partial \omega_{NN}/\partial r|_{\substack{r=R- \\ \theta=\pi/2}}$ is the derivative of the numerical solution for ω_{NN} and can be evaluated by a forward difference scheme.

Equations (5.46) and (5.49) offer two possible methods for evaluating the constant B*. The constant, however, can only be determined by an iterative procedure with an initial guess being made for B*. We chose B* = -1.5 as our initial guess, which is based on the Newtonian solution. The exact iterative procedure which was used to solve the differential equation (4.27) is described in Fig. 7. The reference to 'update ' in this figure refers to the use of equation (5.46) or (5.49). However, we were not able to obtain convergence from either of these two methods for calculating B*. It was found that the average of these two equations gave rise to a convergent solution - i.e. we used

$$B^* = \frac{2}{3}\frac{\partial \psi_{NN}}{\partial r}\bigg|_{\substack{r=R- \\ \theta=\pi/2}} + \frac{r^2}{3}\frac{\partial \omega}{\partial r}\bigg|_{\substack{r=R- \\ \theta=\pi/2}} - \frac{1}{2} - 2\log r \qquad (5.50)$$

to calculate B* iteratively.

When the solution has converged the numerical values for ψ_{NN}, ω_{NN} and their derivatives are known for all r. In particular, at r=a. The non-Newtonian drag correction factor can therefore be evaluated from equations (4.109) and (4.110) where the constants β_2 and β_3 are defined by equations (4.75) and (4.76). It appears therefore, that, in order to calculate the drag force, we require the second and third derivative of the stream function at r=a. However, from equation (5.3) we have

$$\omega_{NN}\Big|_{r=a} = \frac{\partial^2 \psi_{NN}}{\partial r^2}\Big|_{r=a} \qquad (5.51)$$

and

$$\frac{\partial \omega_{NN}}{\partial r}\Big|_{r=a} = \frac{\partial^3 \psi_{NN}}{\partial r^3}\Big|_{r=a} \qquad (5.52)$$

Therefore, the drag coefficient can be evaluated in terms of $\omega_{NN}|_{r=a}$ and $\partial\omega_{NN}/\partial r|_{r=a}$

5.4 Results

5.4.1 Results for the Case when the Outer Sphere has a Finite Radius

In order to test the program validity, the drag force on the inner sphere was produced for the case of a Newtonian fluid when the outer sphere had a finite radius. This drag force is given theoretically by equation (4.97) where d is defined to be the ratio of the radii of the inner and outer spheres. For values of d between 2 and 100, the numerical value obtained for the drag force on the inner sphere agreed to within 0.5% with the value calculated using equation (4.97). In all cases, this accuracy could be achieved using a finite element mesh having 20 elements in the θ direction and 40 elements radially.

The program was modified for the case of a slightly power law fluid when the outer sphere had a finite radius. In this case, the drag correction factor F(n,d) was calculated numerically for d=2, 10, 40 and 100. When d was 100, a finite element mesh with 20 elements in the θ direction and 100 elements radially was used to obtain a stable value for F(n,d). The values are given in Fig. 8 and Table 1. In Fig. 8 the results of Lockyer, Davies and Jones[37] (1980) are also presented. These results have been obtained by means of a finite difference method. In order to approximate the flow situation when the outer sphere has an infinite radius, a value of d significantly greater than 100 would need to be used. In both the finite element and the finite difference method, this is impractical due to rounding errors, demands on the computer memory and running costs. Therefore to overcome these difficulties, a matching technique was developed.

5.4.2 Results for the Case when the Outer Sphere has an Infinite Radius

In order to check our program, the drag force on the inner sphere, when the fluid was Newtonian and the outer sphere had an infinite radius, was obtained numerically. When a finite element mesh having 20 elements in the direction and 140 elements radially was used this value was within 1% of the drag force value calculated using Stoke's law. Approximately 4000 iterations were required for a convergent solution.

The drag correction factor F(n) was then calculated for the case when the fluid was slightly power law and the outer sphere had an infinite radius. Results were obtained for the case when the matching surface had a radius of 10 times and 20 times the radius of the inner sphere and are presented in Figs. 9 and 10 and Tables 2 and 3.

For a grid having 20 elements in the θ direction and 100 elements in the radial direction and using a matching radius of 10 times that of the inner sphere we obtained a value of

$$F'(n) = -1.49911 .$$

and hence F(n) is given by equation (4.109) as

$$F(n) = 1 - 1.49911\ (n-1)$$

It is evident from Fig. 9 that the value for F'(n) has not reached equilibrium and it is likely that more radial elements are required. It appears from Fig. 9 that the value of F'(n) for a mesh having 20 elements in the θ direction, 70 radial elements and a matching radius

10 times that of the inner sphere differs very little from the value F'(n) for a mesh having 20 elements in the θ direction, 140 radial elements and a matching radius 20 times the radius of the inner sphere. This implies that F'(n) is independent of the matching radius chosen for a given density of the finite elements.

5.5 Discussion

In Fig. 11 we present a comparison of our expression in equation (5.54) with the drag correction factor predictions of other workers (Chhrabra[36] (1985)). Most of the theoretical curves 1-7 were produced using a variational principle method. In 1980 Lockyer et al[37] solved the slow flow of a sphere moving through a power law fluid using numerical techniques. However, the analysis was restricted to power law constants near unity and the use of a finite outer sphere. This result is presented in curve 8. Using finite element techniques Tanner[31] (1985) extended the work of Lockyer to the case where the theory was valid for any power law index n (see curve 9). However, the work was carried out for a finite outer sphere with an outer to inner diameter ratio of 20 to 1. A similar approach was used by Crochet[30] (1984) who considered the solution for a sphere moving inside a cylinder with a diameter ratio of 50 to 1 (see curve 10). Although the work of Tanner and Crochet was valid for any power law index n it was, however, restricted to the use of a finite outer boundary and a direct comparison is not possible with curves 1 to 7 which were produced for the flow of a sphere in an infinite expanse of liquid. The work considered in this chapter, is valid for the case of an infinite outer sphere and is presented in curve 11. We also present the drag correction factor for a finite outer sphere, d=100, obtained by using the finite element technique (curve 12) and good agreement was obtained with the result of Lockyer in curve 8.

Since our analysis is only valid for a power law index near unity, then we can also make a comparison with the work of Tanner and Crochet, when n is near unity. Assuming that the drag coefficient is linear with n in the region 0.9 to 1.0 then F'(n) produced by Tanner has a value of -1.4, which is of lower magnitude than our infinite outer sphere value of -1.4991 as expected. We also note that for all values of n, the curve produced by Crochet lies above that produced by Tanner. The result is to be expected since Crochet has chosen an outer cylinder to inner sphere diameter ratio of 50 to 1, in comparison with the outer sphere to inner sphere diameter ratio of 20 to 1 used by Tanner.

The oil drilling industry relies heavily on the ability of an oil drilling fluid to transport rock cuttings from the bottom of an oil well to the surface. This is obviously important for good hole cleaning properties, since a high settling velocity can lead to drilling problems. It is therefore important to be able to accurately predict the settling velocity of a particle moving through an oil drilling fluid. It has been shown (Reynolds[23] (1982) - private communication), that the settling velocity of spherical particles provide an adequate prediction of the settling velocity of rock particles of the same mass and volume. The work considered in this chapter will obviously be a useful contribution to the understanding of the mechanism necessary for the transport of rock cuttings.

CHAPTER 6

PULSATILE FLUID FLOW THROUGH A STRAIGHT HORIZONTAL PIPE OF CIRCULAR CROSS SECTION

6.1 Introduction

An experimental investigation carried out by Barnes et al[38] (1971) has shown that when a sinusoidal pressure gradient is superimposed on a steady flow of visco-elastic liquid in a straight tube of circular cross section the mean flow rate is increased for a given mean pressure gradient. That is flow enhancement occurs during the pulsatile flow of a visco-elastic liquid through a straight tube.

Flow enhancement associated with pulsatile flow is of considerable industrial interest. For example, some mechanical pumps produce a pulsatile gradient as a side effect. These fluctuations are usually deliberately smoothed out and the liquid is, in effect, pumped under a steady pressure gradient. If, however, flow enhancement occurs during the pumping process it may well be advantageous to dispense with smoothing the flow.

The possibility of deliberately increasing flow rates by pulsing is attractive only if the advantages gained are not offset by the increase in power required to maintain the pulsations. This question has been addressed by Edwards et al[39] (1972) and Phan-Thien and Dudek[40] (1982). These workers have shown that when a power law fluid is used, then there is no economic advantage to be gained in pulsating the flow.

An analysis of the pulsatile flow behaviour has been carried out for several different visco-elastic fluids. These include the second, third and fourth order Oldroyd fluids (Walters and Townsend[41] (1968), Barnes et al (1971)), the Goddard-Miller fluids (Davies et al[42] (1978)) and the generalised Maxwell fluids (Phan-Thien[43,44] (1978, 1980)). The experimental results of Barnes et al show that the mean flow rate increased with pulsatile frequency when the ratio of the pulsatile pressure amplitude to the steady pressure gradient was held constant. These results are in contradiction with the theoretical predictions obtained by Walters and Townsend and Davies et al. They are, however, in some agreement with the theoretical predictions of Phan-Thien.

A conventional apparatus was used by Barnes et al in which the mean pressure gradient was controlled and the mean flow rate was measured. With this arrangement whenever a comparison of the experimental results is made with theory the amplitude of the pulsatile pressure gradient must be measured. This measurement is difficult to obtain accurately and can, therefore, be subject to error. This is particularly true at high frequencies due to the greater effect of inertia in the transducer connecting tube. An alternative apparatus has been constructed by Chakrabati and Davies[45] (1980) who have carried out experimental work in which the mean flow rate and the pulsatile flow rate were controlled and the pressure gradient was measured. Their apparatus is shown, schematically, in Fig. 12.

The following points regarding their experiment are worth noting:

(1) a known steady flow rate was controlled by the constant movement of a piston in a resevoir of an Instron machine and so no smoothing was necessary.

(2) the oscillatory device took the form of a moving oscillatory piston in a sealed chamber and therefore the oscillatory flow rate was known exactly.

(3) the pressure transducer was activated by the pressure of a light, low viscosity oil in the connecting tubes. This light oil was able to faithfully follow the pressure variations in the straight tube. The pressure amplitude measurement, however, is not required for the comparison of experimental results with theoretical predictions but can be used as a check for the consistency of data.

It has been shown by Davies et al[42] (1978) that flow enhancement is strongly dependent on the shape of the viscosity - shear rate curve. Therefore it is important to choose a model which has the correct viscosity behaviour. The Goddard-Miller model used by these authors had a power law viscosity function to predict flow enhancement behaviour but the theory was restricted to small oscillatory amplitudes. Townsend[46] (1973) has carried out his analysis using an Oldroyd four constant model, the theory being valid for any oscillatory pressure amplitude. The Oldroyd model, however, does not give a realistic viscosity function. In this Chapter we shall be concerned with an inelastic power law fluid analysis which is not restricted to small amplitudes and which also includes the effects of fluid inertia. The relevant equations of motion will be solved numerically. A perturbation analysis was also carried to provide analytical expressions for the flow enhancement when the effects of fluid inertia are small.

6.2 Pulsatile Flow Analysis

6.2.1 Basic Equations

The numerical analysis given here initially follows the work of Edwards et al[39] (1972). As a first check against the validity of the numerical procedures we rederived the numerical results of these authors. The program was then modified to allow for the case when the pulsed flow rate behaviour is known.

A cylindrical polar co-ordinate system (r, θ, z) was used with the positive z-axis in the direction of flow. It was assumed that the length of the tube was large compared to its radius and, consequently, edge effects were ignored.

A velocity field of the form

$$V_r = 0 \ , \quad V_\theta = 0 \ , \quad V_z = w(r, t) \qquad (6.1)$$

was assumed. This velocity field automatically satisfies the equation of continuity. When body forces are omitted the equations of motion become

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \, \tau_{rz} \right) \right]$$

$$\frac{\partial p}{\partial r} = 0 \qquad (6.2)$$

$$\frac{\partial p}{\partial \theta} = 0$$

where $$\tau_{rz} = -K \dot{\gamma}^{\,n-1} \dot{\gamma}_{rz} \qquad (6.3)$$

Here $\dot{\gamma}_{ij}$ is the rate of strain tensor and τ_{ij} is the stress tensor. From equations (6.1) and (6.3) we obtain

$$\tau_{rz} = -k\left|\frac{\partial w}{\partial r}\right|^{n-1}\frac{\partial w}{\partial r} \qquad (6.4)$$

and the equation of motion becomes

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(rk\left|\frac{\partial w}{\partial r}\right|^{n-1}\frac{\partial w}{\partial r}\right) \qquad (6.5)$$

where $\partial p/\partial z$ is a function of t only.

6.2.2 The Case for Steady Flow

When the flow is steady $w=w(r)$ only and $\partial w/\partial t=0$. The equation of motion becomes

$$r\frac{\partial p}{\partial z} = k\frac{\partial}{\partial r}\left(r\left(-\frac{\partial w}{\partial r}\right)^{n-1}\frac{\partial w}{\partial r}\right) \qquad (6.6)$$

Integrating equation (6.6) with respect to r we obtain

$$\frac{r}{2}\frac{\partial p}{\partial z} = -k\left(-\frac{\partial w}{\partial r}\right)^{n} \qquad (6.7)$$

where $-\partial w/\partial r$ is positive. The mean flow rate, Q, is given by

$$Q = \pi a^2 \bar{u} = \int_0^a 2\pi r w \, dr \qquad (6.8)$$

where $\bar{u}$ is the mean velocity of the fluid and a is the radius of the circular pipe. Integrating by parts and using the no slip condition at the pipe walls we obtain

$$a^2 \bar{u} = \int_0^a r^2 \left(- \frac{\partial w}{\partial r} \right) dr \qquad (6.9)$$

From equations (6.7) and (6.9) using $\partial p / \partial r = 0$

$$\bar{u} = \frac{1}{a^2} \left(\frac{1}{2K} \right)^{\frac{1}{n}} \left(- \frac{\partial p}{\partial z} \right)^{\frac{1}{n}} \int_0^a r^{2+\frac{1}{n}} \, dr \qquad (6.10)$$

and so

$$\frac{\partial p}{\partial z} = \frac{- \bar{u} \, 2K}{a^{n+1}} \left(\frac{3n+1}{n} \right)^n \qquad (6.11)$$

we shall denote the steady pressure gradient by $-p_s$ and so

$$p_s = \frac{2 \bar{u}^n K}{a^{n+1}} \left(\frac{3n+1}{n} \right)^n \qquad (6.12)$$

6.2.3 The Case for Pusatile Flow

In the case of pulsatile flow we let

$$\frac{\partial p}{\partial z} = - p_s \left(1 + \epsilon \cos \omega t \right) \qquad (6.13)$$

where ω is the frequency in radians per second and ϵ is the ratio between the pulsed pressure ~~flow rate~~ amplitude and the steady pressure ~~flow rate~~ gradient ~~amplitude~~. The pressure gradient is now a function of time i.e.

$$- \frac{\partial p}{\partial z} = P(t) \qquad (6.14)$$

and the equation of motion becomes

$$\rho \frac{\partial w}{\partial t} = P(t) - \frac{1}{r} \frac{\partial}{\partial r} \left(r K \left(\frac{\partial w}{\partial r} \right) \left| \frac{\partial w}{\partial r} \right|^{n-1} \right) \qquad (6.15)$$

The equation can now be non-dimensionalised using the following changes of variable

$$T = \frac{\omega t}{2\pi}, \quad R = \frac{r}{a}, \quad V = \frac{w}{\bar{u}} \tag{6.16}$$

to give

$$\bar{u}\rho \frac{W}{2\pi}\frac{\partial V}{\partial T} = P(T) + \frac{K\bar{u}^n}{a^{n+1}} \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \tag{6.17}$$

where

$$P(T) = \frac{\bar{u}^n}{a^{n+1}} 2K\alpha P^*(T) \tag{6.18}$$

$$P^*(T) = 1 + \epsilon \cos 2\pi T \tag{6.19}$$

and

$$\alpha = \left(\frac{3n+1}{n}\right)^n \tag{6.20}$$

Thus

$$\left(\frac{a^{n+1}}{\bar{u}^n K}\omega\rho\frac{\bar{u}}{2\pi}\right)\frac{\partial V}{\partial T} = 2\alpha P^* + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \tag{6.21}$$

Defining a Reynold's number by

$$Re = 8\left(\frac{n}{3n+1}\right)\rho\frac{\bar{u}^{2-n}a^n}{K} \tag{6.22}$$

we have

$$\frac{F\alpha}{8}\frac{\partial V}{\partial T} = 2\alpha P^*(T) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \tag{6.23}$$

where $F = \dfrac{a\omega Re}{2\pi\bar{u}}$ is dimensionless (6.24)

Equation (6.23) has to be solved subject to the boundary conditions

$$\begin{array}{lll} \text{(i)} & V(1,T) = 0 & T \geqslant 0 \\ \text{(ii)} & \left.\dfrac{\partial V}{\partial R}\right|_{R=1} = 0 & T \geqslant 0 \\ \text{(iii)} & V\big|_{T=0} = 0 & 0 \leqslant R \leqslant 1 \end{array} \tag{6.25}$$

These boundary conditions correspond to the physical conditions

(i) there is no slip at the pipe wall

(ii) the fluid shear rate along the tube axis is zero

(iii) the fluid is considered, initially, to be at rest

6.2.4 The Numerical Method for the Case when the Pressure Gradient is known

Equation (6.23) was solved using the explicit difference scheme employed by Edwards et al (1972). The domain was divided by intervals ΔR and ΔT, the velocity at any point being $V_{I,J}$ where I indicates the radial position and J the time step. The radius, $R = 0$ to $R = 1$, was divided into N-1 intervals giving $\Delta R = 1/(N-1)$.

In order to improve the numerical accuracy of the scheme we follow Edwards et al and use forward and backward differences as follows

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)$$

$$\simeq \frac{1}{R\Delta R}\left[(R+\Delta R)\left.\frac{\partial V}{\partial R}\right|_{R+\Delta R}\left|\frac{\partial V}{\partial R}\right|^{n-1}_{R+\Delta R} - R\left.\frac{\partial V}{\partial R}\right|_{R}\left|\frac{\partial V}{\partial R}\right|^{n-1}_{R}\right] \quad (6.26)$$

And

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)$$

$$\simeq \frac{1}{R\Delta R}\left[R\left.\frac{\partial V}{\partial R}\right|_{R}\left|\frac{\partial V}{\partial R}\right|^{n-1}_{R} - (R-\Delta R)\left.\frac{\partial V}{\partial R}\right|_{R-\Delta R}\left|\frac{\partial V}{\partial R}\right|^{n-1}_{R-\Delta R}\right] \quad (6.27)$$

Equation (6.26) was derived using a backward difference approximation for $\partial V/\partial R$ while equation (6.27) was derived using the corresponding forward difference aproximation. The two approximations given by the equations were then averaged to give

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)$$

$$\simeq \frac{1}{R(\Delta R)^{n-1}}\Big[\left(R+\frac{\Delta R}{2}\right)\left(V_{I+1,J}-V_{I,J}\right)\left|V_{I+1,J}-V_{I,J}\right|^{n-1}$$

$$-\left(R-\frac{\Delta R}{2}\right)\left(V_{I,J}-V_{I-1,J}\right)\left|V_{I,J}-V_{I-1,J}\right|^{n-1} \qquad (6.28)$$

A forward difference was used for $\partial V/\partial T$

$$\left(\frac{\partial V}{\partial T}\right)_{I,J} = \frac{1}{\Delta T}\left(V_{I,J+1}-V_{I,J}\right) \qquad (6.29)$$

Substituting equations (6.28) and (6.29) into equation (6.23) we have an explicit finite difference scheme for $V_{I,J+1}$ in terms of $V_{I-1,J}$, $V_{I,J}$ and $V_{I+1,J}$, valid except when I = 1 and I = N. These values of I correspond to the pipe centreline (R = 0) and the pipe wall (R = 1)

On the pipe wall, by boundary condition (i) of equation (6.25)

$$V_{N,J} = 0 \qquad J \geq 1 \qquad (6.30)$$

On the pipe centreline equation (6.28) has a singularity since R = 0. This singularity is removed using L'Hopitals theorem. By this theorem

$$\lim_{R\to 0}\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) = 2\frac{\partial}{\partial R}\left(\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)\Bigg|_{R=0} \qquad (6.31)$$

Using a forward difference approximation for $\partial V/\partial R|\partial V/\partial R|^{n-1}$ and a backward difference approximation for $2\partial/\partial R$ we have

$$\lim_{R\to 0}\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)$$

$$\simeq \frac{2}{(\Delta R)^{n+1}}\Big[\left(V_{I+1,J}-V_{I,J}\right)\left|V_{I+1,J}-V_{I,J}\right|^{n-1}$$

$$-\left(V_{I,J}-V_{I-1,J}\right)\left|V_{I,J}-V_{I-1,J}\right|^{n-1}\Big] \qquad (6.32)$$

From boundary condition (ii) of equation (6.25)

$$V_{I+1,J} = V_{I-1,J} \tag{6.33}$$

when I = 1. Substituting equation (6.33) into equation (6.32) we obtain

$$\lim_{R\to 0} \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \simeq \frac{4}{(\Delta R)^{n+1}}\left(V_{2,J}-V_{1,J}\right)\left|V_{2,J}-V_{1,J}\right|^{n-1} \tag{6.34}$$

Thus $V_{I,J}$ can now be approximated numerically for $1 \leqslant I \leqslant N$ and $1 \leqslant J \leqslant U$ where U is some fixed upper limit for J.

To confirm the correctness of the program the results of Edwards et al were rederived.

6.2.5 Modifying the Numerical Method for the Case when the Flow Rate is known

Equation (6.23) was now modified and expressed in terms of the flow rate rather than the pressure gradient since in our case it is the flow rate, and not the pressure gradient, which is known as a function of time. Using equation (6.16) the flow rate from equation (6.8) can be expressed as

$$Q = \int_0^1 2\pi a^2 \bar{u}\, V R\, dR \tag{6.35}$$

Therefore

$$\frac{\partial Q}{\partial T} = 2\pi a^2 \bar{u} \int_0^1 \frac{\partial V}{\partial T} R\, dR \tag{6.36}$$

Integrating equation (6.23) with respect to R and using equation (6.36) we have

$$\frac{F\alpha}{8}\frac{\partial Q}{\partial T} = 4\pi\alpha a^2\bar{u}\left[\frac{R^2}{2}\right]_0^1 P^*(T) + 2\pi a^2\bar{u}\left[R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right]_0^1 \tag{6.37}$$

We define a non-dimensional flow rate Q* by

$$Q = \pi a^2 \bar{u} \, Q^* \qquad (6.38)$$

Substituting equation (6.38) into equation (6.37) we obtain

$$\frac{F}{16}\frac{\partial Q^*}{\partial T} = P^*(T) + \frac{1}{\alpha}\left(\frac{\partial V}{\partial R}\right)_{R=1}\left|\frac{\partial V}{\partial R}\right|^{n-1}_{R=1} \qquad (6.39)$$

Eliminating the unknown pressure gradient from equations (6.39) and (6.23) we obtain

$$\frac{F\alpha}{8}\frac{\partial V}{\partial T} = \frac{F\alpha}{8}\frac{\partial Q^*}{\partial T} - 2\left(\frac{\partial V}{\partial R}\right)_{R=1}\left|\frac{\partial V}{\partial R}\right|^{n-1}_{R=1}$$

$$+ \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \qquad (6.40)$$

where experimentally it is known that

$$Q^* = 1 + \epsilon \cos 2\pi T \qquad (6.41)$$

and so

$$\frac{\partial Q^*}{\partial T} = -2\pi\epsilon \sin 2\pi T \qquad (6.42)$$

Equation (6.42) when substituted into (6.40) gives an equation which can be solved to give the velocity-profile at time T. Hence we can numerically determine $(\partial V/\partial R)_{R=1} \, |\partial V/\partial R|^{n-1}_{R=1}$ which on substitution into equation (6.39) will give the value of $P^*(T)$.

We used the same finite difference scheme to solve equation (6.40) as we used to solve equation (6.23). However equation (6.40) is insufficient to determine the V-profile uniquely since the steady flow rate component is not considered. To overcome this the V-profile was adjusted at the end of each cycle to give a new V-profile consistent with Q^*. The new V-profile was expressed in terms of the original V-profile by

$$V(R)_{NEW} = V(R)_{ORIG} + C(R) \qquad (6.43)$$

where C(R) is a correction term. At the pipe wall, i.e. when R=1, the correction term must be zero since V is held zero here by the no slip condition. A maximum error of λ was assumed at the central axis of the pipe, i.e. when R=0, and C(R) was taken to be linear in R. Thus

$$C(R) = \lambda(1 - R) \tag{6.44}$$

Substituting equation (6.38) into equation (6.35) gives

$$Q^* = \int_0^1 2V(R)R\,dR \tag{6.45}$$

Thus, at the end of each cycle we require that

$$Q^* = \int_0^1 2V(R)_{NEW} R\,dR \tag{6.46}$$

On substituting equation (6.43) and (6.44) into equation (6.46) we obtain

$$Q^* = \int_0^1 2V(R)_{ORIG} R\,dR + \frac{\lambda}{3} \tag{6.47}$$

which may be written more concisely as

$$Q^* = Q_{NUM} + \frac{\lambda}{3} \tag{6.48}$$

giving

$$\lambda = 3(Q^* - Q_{NUM}) \tag{6.49}$$

The adjustment of the velocity-profile was carried out at a number of different time steps, from one time step to the full cycle.

When using equation (6.39) the normalised pressure gradient P*(T) was calculated for each T step and it was found that, under certain circumstances, numerical instabilities were present. These instabilities gave rise to alternating sign changes in the pressure values. A routine was incorporated into the program to count the number of sign alterations in each cycle. It was found that repeatable data could be obtained provided that not more than 5% of the data suffered sign changes. In most cases, the instability could either be removed or reduced to give repeatable results by decreasing Δ T. For some combinations of N, ϵ and F, however, it was not possible to obtain convergence of the solution. Such cases are indicated in Tables 4 and 5 It was found that for stable, or sufficiently stable, solutions that no more than three cycles were required before flow became fully developed and independent of the starting condition.

Finally, a Fourier analysis was carried out over the values of $P^*(T)$ in the third cycle. The fundamental amplitude and higher harmonics were calculated.

The following results were plotted

(i) The variation of the normalised mean pressure gradient with F with N in the range 0.3 to 1.0 for given ϵ. The value of ϵ being 0.5, 1, 5 or 10. The results are plotted in Figs. 13 - 16 and the values are tabulated in Tables 4 - 7.

(ii) As for (i) but the phase angle variation was considered. The results are plotted in Figs. 17 - 20 and the values are tabulated in Tables 13 - 16.

(iii) As for (i) but the amplitude variation of the pulsed flow was considered. The results are plotted in Figs. 21 - 24 and the values are tabulated in Tables 17 - 20.

(iv) The normalised mean pressure gradient against ϵ was considered with F in the range 5 to 100, N being held constant at 0.5. The results are plotted in Fig. 25 and the values are tabulated in Table 21.

In order to check the correctness of the pressure gradient data, we substituted these values as input data to our original program. The flow rate output data gave good agreement with the known flow rate waveform. In order to check the case when F is small (i.e. when fluid inertia is small) a perturbation analysis was carried out which we shall now consider.

6.3 Pulsatile Analysis for Small Fluid Inertia

6.3.1 Expressing the Flow Rate in Terms of the Pressure Gradient

A velocity distribution of the form

$$w(r,t) = w_0(r,t) + \rho\, w_1(r,t) + \rho^2 w_2(r,t) + \cdots \quad (6.50)$$

was assumed. The boundary conditions which must be satisfied are

$$W_0(a,t) = W_1(a,t) = W_2(a,t) = 0, \quad t \geq 0 \tag{6.51}$$

and

$$\frac{\partial W_0}{\partial r}(0,t) = \frac{\partial W_1}{\partial r}(0,t) = \frac{\partial W_2}{\partial r}(0,t) = 0, \quad t \geq 0 \tag{6.52}$$

Substituting equation (6.50) into equation (6.8) gives

$$Q(t) = Q_0(t) + \rho Q_1(t) + \rho^2 Q_2(t) + \cdots \tag{6.53}$$

Throughout analysis terms of order ρ^3 and higher will be ignored.

We have from equation (6.16) that $V = w/\bar{u}$. Substituting this into equation (6.50) gives

$$V(r,t) = V_0(r,t) + \rho V_1(r,t) + \rho^2 V_2(r,t) + \cdots \tag{6.54}$$

Substituting V(r,t) from equation (6.54) into equation (6.15) and using Taylor's expansion gives

$$\bar{u}\left(\rho \frac{\partial V_0}{\partial t} + \rho^2 \frac{\partial V_1}{\partial t}\right) = P(t) + \frac{k}{r}\frac{\partial}{\partial r}\left[r\left|\frac{\partial V_0}{\partial r}\right|^{n-1}\left(\frac{\partial V_0}{\partial r} + \rho n \frac{\partial V_1}{\partial r}\right.\right.$$
$$\left.\left. + \rho^2 n\left(\frac{\partial V_2}{\partial r} + \frac{n-1}{2}\frac{(\partial V_1/\partial r)^2}{\partial V_0/\partial r}\right)\right)\right]\bar{u}^n \tag{6.55}$$

The coefficients of ρ^0, ρ^1 and ρ^2 are now compared as follows:

Coefficient of ρ^0

From equation (6.55)

$$P(t) + \bar{u}^n \frac{k}{r}\frac{\partial}{\partial r}\left(r\left|\frac{\partial V_0}{\partial r}\right|^{n-1}\frac{\partial V_0}{\partial r}\right) = 0 \tag{6.56}$$

Multiplying through by r and integrating with respect to r we obtain

$$r\left|\frac{\partial V_0}{\partial r}\right|^{n-1}\frac{\partial V_0}{\partial r} = -\frac{r^2}{2K}\frac{P(t)}{\bar{u}^n} + A(t) \tag{6.57}$$

where A(t) is an arbitrary function of time.
From equations (6.52) we have

$$A(t) = 0 \qquad (6.58)$$

Rearranging equation (6.57) and using equation (6.58) we obtain

$$\frac{\partial V_0}{\partial r} = \frac{-r^{1/n}}{(2K)^{\frac{1}{n}}\bar{u}} P|P|^{\frac{1}{n}-1} \qquad (6.59)$$

which allows for both positive and negative values of P(t). Integrating equation (6.59) with respect to r and satisfying the boundary condition

$$V_0(a\ t) = 0 \ , \quad t \geqslant 0 \qquad (6.60)$$

we obtain

$$V_0(r,t) = \frac{n\,\bar{u}^{-1}}{(2K)^{1/n}(n+1)}\left[a^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right] P|P|^{\frac{1-n}{n}} \qquad (6.61)$$

On substituting V = w/ū into equation (6.8)and integrating by parts we obtain

$$Q(t) = -\pi\int_0^a \bar{u}\, r^2 \frac{\partial V}{\partial r}\, dr \qquad (6.62)$$

Substituting equation (6.61) into equation (6.62) and integrating we obtain

$$Q_0(t) = \frac{\pi n\, a^{\frac{3n+1}{n}}}{(2K)^{1/n}(3n+1)} P|P|^{\frac{1-n}{n}} \qquad (6.63)$$

Coefficient of ρ

Comparing the first order coefficient in ρ in equation (6.55) we obtain

$$\bar{u}\,\frac{\partial V_0}{\partial t} = \frac{Kn}{r}\,\frac{\partial}{\partial r}\left(r\left|\frac{\partial V_0}{\partial r}\right|^{n-1}\frac{\partial V_1}{\partial r}\right)\bar{u}^{\,n} \qquad (6.64)$$

Substituting equation (6.61) into equation (6.64) and integrating with respect to r we obtain

$$\frac{\partial V_1}{\partial r} = \frac{2}{(2K)^{2/n}(n+1)\bar{u}}\left[\frac{r^{1/n}a^{\frac{1+n}{n}}}{2} - \frac{n\,r^{1+2/n}}{3n+1}\right]|P|^{\frac{1-n}{n}}\frac{d}{dt}\left(P|P|^{\frac{1-n}{n}}\right)$$
$$+ B(t)\,r^{-2+\frac{1}{n}} \qquad (6.65)$$

The boundary condition which must be satisfied is given by equation (6.52)

When n > 1/2 B(t) must be zero, but when n < 1/2 then B(t) can remain arbitrary and it seems that a further boundary condition is required in order to obtain a unique solution in this case. However, the power low fluid predicts an infinite viscosity when the shear rate is zero. In general at low shear rates power low fluids behave as Newtonian fluids and when n = 1 (the Newtonian case) then the last term in equation (6.65) is B(t)/r and B(t) must be zero. Hence B(t) is zero for all values that n may take.

On integrating equation (6.65) with respect to r and then satisfying the boundary condition $V_1(a,t) = 0$ we obtain

$$V_1(r,t) = \frac{n}{\bar{u}(2K)^{2/n}(n+1)^2}\left[r^{1+\frac{1}{n}}a^{1+\frac{1}{n}} - \frac{n\,r^{2+\frac{2}{n}}}{3n+1} - \left(\frac{2n+1}{3n+1}\right)a^{2+\frac{2}{n}}\right]$$
$$|P|^{\frac{1-n}{n}}\frac{d}{dt}\left(P|P|^{\frac{1-n}{n}}\right) \qquad (6.66)$$

On substituting $V_1(r,t)$ from equation (6.66) into

$$Q_1(t) = -\pi\int_0^a \bar{u}\,r^2\,\frac{\partial V_1}{\partial r}\,dr \qquad (6.67)$$

we obtain

$$Q_1(t) = \frac{\pi a^{4+\frac{2}{n}} n}{(2K)^{2/n}(3n+1)(2n+1)} |P|^{\frac{1}{n}-1} \frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right) \qquad (6.68)$$

i.e.

$$Q_1(t) = \frac{-\pi a^{4+\frac{2}{n}} n}{(2K)^{2/n}(3n+1)(2n+1)} \frac{|P|^{1/n}}{P} \frac{d}{dt}\left(|P|^{\frac{1}{n}}\right) \qquad (6.69)$$

<u>Coefficient of ρ^2</u>

Comparing terms of order ρ^2 in equation (6.55) we have

$$\frac{\partial V_1}{\partial t} = \frac{Kn}{r}\frac{\partial}{\partial r}\left[r\left|\frac{\partial V_0}{\partial r}\right|^{n-1}\left(\frac{\partial V_2}{\partial r} + \frac{n-1}{2}\frac{(\partial V_1/\partial r)^2}{\partial V_0/\partial r}\right)\right]\bar{u}^{-n-1} \qquad (6.70)$$

Substituting V_0 from equation (6.61) and V_1 from equation (6.66) into equation (6.70) and integrating with respect to r we obtain

$$\frac{\partial V_2}{\partial r} = \frac{2(n-1)\bar{u}^{-1}}{(2K)^{3/n}(n+1)^2}\left[\frac{r^{1/n}a^{2+2/n}}{4} - \frac{n r^{1+2/n} a^{1+1/n}}{3n+1} + \frac{n^2 r^{2+3/n}}{(3n+1)^2}\right]$$

$$\cdot \frac{|P|^{\frac{1}{n}-1}}{P}\left(\frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right)\right)^2$$

$$+ \frac{2\bar{u}^{-1}}{(2K)^{3/n}(n+1)^2(3n+1)}\left[n r^{1+\frac{2}{n}} a^{1+\frac{1}{n}} - \frac{n^2 r^{2+3/n}}{2(2n+1)} - \frac{a^{2+2/n}(2n+1) r^{\frac{1}{n}}}{2}\right]$$

$$\cdot |P|^{1/n}\frac{d}{dt}\left[|P|^{\frac{1}{n}-1}\frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right)\right] \qquad (6.71)$$

where the constant of integration is zero for reasons previously discussed.

The flow rate $Q_2(t)$ is given by substituting $\partial V_2/\partial r$ from equation (6.71) into

$$Q_2(t) = -\pi\bar{u}\int_0^a r^2\frac{\partial V_2}{\partial r}\,dr \qquad (6.72)$$

to give

$$Q_2(t) = \frac{\pi n(8n+3)a^{\frac{5n+3}{n}}}{2(2K)^{3/n}(3n+1)(5n+3)(2n+1)}\left[-\frac{(n-1)}{P}|P|^{\frac{1}{n}-1}\left(\frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right)\right)^2 \right.$$
$$\left. + 2|P|^{\frac{1}{n}-1}\frac{d}{dt}\left(|P|^{\frac{1}{n}-1}\frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right)\right)\right] \qquad (6.73)$$

Equation (6.73) simplifies to

$$Q_2(t) = \frac{\pi n(8n+3)a^{\frac{5n+3}{n}}}{(2K)^{3/n}(3n+1)^2(5n+3)(2n+1)}\left[-\frac{3}{2}(n-1)\frac{|P|^{\frac{1}{n}-1}}{P}\left(\frac{d}{dt}\left(|P|^{\frac{1}{n}}\right)\right)^2 \right.$$
$$\left. + \frac{|P|^{2/n-1}}{P}\frac{d^2}{dt^2}\left(|P|^{1/n}\right)\right] \qquad (6.74)$$

On carrying out the differentiation it can be seen that when $P(t)=0$ then $Q_2(t)$ becomes infinite for $3/4 < n < 1$.

The final expression for $Q(t)$ is obtained by substituting the equation for Q_0, Q_1 and Q_2 derived above into equation (6.53) to give

$$Q(t) = \frac{\pi n a^{\frac{3n+1}{n}}}{(2K)^{1/n}(3n+1)} P|P|^{\frac{1}{n}-1} - \frac{\rho\pi a^{\frac{4n+2}{n}} n|P|^{1/n}}{(2K)^{2/n}(3n+1)(2n+1)P}\frac{d}{dt}\left(|P|^{\frac{1}{n}}\right)$$
$$+ \frac{\rho^2\pi n(8n+3)a^{\frac{5n+3}{n}}}{(2K)^{3/n}(3n+1)^2(5n+3)(2n+1)}\left[-\frac{3}{2}(n-1)\frac{|P|^{1/n-1}}{P}\left(\frac{d}{dt}\left(|P|^{1/n}\right)\right)^2 \right.$$
$$\left. + \frac{|P|^{2/n-1}}{P}\frac{d^2}{dt^2}\left(|P|^{1/n}\right)\right] \qquad (6.75)$$

The mean flow rate is given by

$$\langle Q\rangle = \langle Q_0\rangle + \langle Q_1\rangle + \langle Q_2\rangle \qquad (6.76)$$

where $\langle\ \rangle$ denotes the time average value, over one cycle of oscillation of the expression contained within the brackets. Hence

$$\langle Q_0\rangle = \frac{\pi n a^{\frac{3n+1}{n}}}{(2K)^{1/n}(3n+1)}\langle P|P|^{\frac{1}{n}-1}\rangle \qquad (6.77)$$

$$\langle Q_1 \rangle = \frac{-\rho \pi a^{\frac{4n+2}{n}} n}{(2K)^{2/n}(3n+1)(2n+1)} \left\langle \frac{|P|^{1/n}}{P} \frac{d}{dt}\left(|P|^{1/n}\right) \right\rangle \tag{6.78}$$

and

$$\langle Q_2 \rangle = \frac{\rho^2 \pi n (8n+3) a^{\frac{5n+3}{n}}}{(2K)^{3/n}(3n+1)^2(5n+3)(2n+1)} \left[\left\langle -\frac{3}{2}(n-1) \frac{|P|^{\frac{1}{n}-1}}{P} \left(\frac{d}{dt}\left(|P|^{\frac{1}{n}}\right)\right)^2 \right\rangle \right.$$

$$\left. + \left\langle \frac{|P|^{2/n-1}}{P} \frac{d^2}{dt^2}\left(|P|^{1/n}\right) \right\rangle \right] \tag{6.79}$$

The expression for $\langle Q_2 \rangle$ can now be simplified by using integration by parts on the first term. We obtain

$$\int \frac{|P|^{\frac{1}{n}-1}}{P} \frac{d}{dt}\left(|P|^{1/n}\right) dt = \int \frac{|P|^{\frac{1}{n}-1}}{P} \cdot \frac{1}{n} \frac{|P|^{\frac{1}{n}}}{P} \frac{dP}{dt}\, dt \tag{6.80}$$

$$= \frac{1}{n} \frac{|P|^{\frac{2}{n}-2}}{\left(\frac{2}{n}-2\right)} \frac{|P|}{P} \tag{6.81}$$

i.e.

$$\int \frac{|P|^{\frac{1}{n}-1}}{P} \frac{d}{dt}\left(|P|^{\frac{1}{n}}\right) dt = \frac{1}{2(1-n)} \frac{|P|^{2/n-1}}{P} \tag{6.82}$$

Therefore

$$\left\langle -\frac{3}{2}(n-1) \frac{|P|^{\frac{1}{n}-1}}{P} \frac{d}{dt}\left(|P|^{\frac{1}{n}}\right) \frac{d}{dt}\left(|P|^{1/n}\right) \right\rangle$$

$$\doteq \frac{3}{2}(n-1) \left\langle \frac{d^2}{dt^2}\left(|P|^{\frac{1}{n}}\right) \frac{1}{2(1-n)} \frac{|P|^{\frac{2}{n}-1}}{P} \right\rangle \tag{6.83}$$

$$= \frac{3}{4} \left\langle \frac{|P|^{2/n-1}}{P} \frac{d^2}{dt^2}\left(|P|^{\frac{1}{n}}\right) \right\rangle \tag{6.84}$$

Adding this to the second component gives

$$\langle Q_2 \rangle = \frac{\rho^2 \pi n (8n+3) a^{\frac{5n+3}{n}}}{4(2K)^{3/n}(3n+1)^2(5n+3)(2n+1)} \left\langle \frac{|P|^{7/n-1}}{P} \frac{d^2}{dt^2}\left(|P|^{1/n}\right) \right\rangle \tag{6.85}$$

6.3.2 Expressing the Pressure Gradient in terms of the Flow Rate

In the experiment carried out by Chakrabati and Davies the flow rate was known and the pressure gradient was measured. Hence, we require to make P(t) the subject of equation (6.75). This was achieved using a Taylor's series expansion as described below.

Equation (6.75) can be written as

$$Q(t) = k_0 P|P|^{1/n-1} + k_1 \rho |P|^{1/n-1} \frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right)$$

$$+ k_2 \rho^2 \left[-(n-1)\frac{|P|}{P}^{\frac{1}{n}-1}\left(\frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right)\right)^2\right.$$

$$\left. + 2|P|^{\frac{1}{n}-1}\frac{d}{dt}\left(|P|^{\frac{1}{n}-1}\frac{d}{dt}\left(P|P|^{\frac{1}{n}-1}\right)\right)\right] \quad (6.86)$$

where

$$k_0 = \frac{\pi n a^{\frac{3n+1}{n}}}{(2K)^{1/n}(3n+1)} \quad (6.87)$$

$$k_1 = \frac{-\pi n a^{\frac{4n+2}{n}}}{(2K)^{2/n}(3n+1)(2n+1)} \quad (6.88)$$

$$k_2 = \frac{\pi n (8n+3) a^{\frac{5n+3}{n}}}{2(2k)^{3/n}(3n+1)^2(5n+3)(2n+1)} \quad (6.89)$$

we allow P(t) to be given by

$$P(t) = P_0(t) + \rho P_1(t) + \rho^2 P_2(t) \quad (6.90)$$

where P_0, P_1 and P_2 are functions of t which must be determined. From equation (6.81) it is evident that we need to evaluate the term $P|P|^{-1+1/n}$ to order ρ^2 and the term $|P|^{-1+1/n}$ to order ρ. From equation (6.90) we have

$$P|P|^{\frac{1}{n}-1} = (P_0 + \rho P_1 + \rho^2 P_2)|P_0 + \rho P_1 + \rho^2 P_2|^{\frac{1}{n}-1} \qquad (6.91)$$

Using a Taylor's series expansion and ignoring terms of order ρ^3 and higher we have

$$P|P|^{\frac{1}{n}-1} = P_0|P_0|^{\frac{1}{n}-1} + \frac{\rho}{n} P_1|P_0|^{\frac{1}{n}-1}$$
$$+ \rho^2\left(\frac{1}{n}P_2|P_0|^{\frac{1}{n}-1} + \frac{1-n}{2n^2}\frac{P_1^2}{P_0}|P_0|^{\frac{1}{n}-1}\right) \qquad (6.92)$$

Similarly the term $|P|^{-1+\frac{1}{n}}$ is given, to order ρ, by

$$|P|^{\frac{1}{n}-1} = |P_0|^{\frac{1}{n}-1} + \rho\frac{1-n}{n}\frac{P_1}{P_0}|P_0|^{\frac{1}{n}-1} \qquad (6.93)$$

Substituting equations (6.90), (6.92) and (6.93) into equation (6.86) we obtain

$$Q(t) = k_0\Big[P_0|P_0|^{\frac{1}{n}-1} + \frac{\rho}{n} P_1|P_0|^{\frac{1}{n}-1}$$
$$+ \rho^2\Big(\frac{1}{n}P_2|P_0|^{\frac{1}{n}-1} + \frac{1-n}{2n^2}\frac{P_1^2}{P_0}|P_0|^{\frac{1}{n}-1}\Big)\Big]$$
$$+ k_1\rho\Big[|P_0|^{\frac{1}{n}-1} + \rho\frac{1-n}{n}\frac{P_1}{P_0}|P_0|^{\frac{1}{n}-1}\Big]$$
$$\cdot\Big[\frac{d}{dt}\Big(P_0|P_0|^{\frac{1}{n}-1} + \frac{\rho}{n}P_1|P_0|^{\frac{1}{n}-1}\Big]$$
$$+ k_2\rho^2\Big[-\frac{n-1}{P_0}|P_0|^{\frac{1}{n}-1}\Big(\frac{d}{dt}\Big(P_0|P_0|^{\frac{1}{n}-1}\Big)\Big)^2$$
$$+ 2|P_0|^{\frac{1}{n}-1}\frac{d}{dt}\Big(|P_0|^{\frac{1}{n}-1}\frac{d}{dt}\Big(P_0|P_0|^{\frac{1}{n}-1}\Big)\Big)\Big] \qquad (6.94)$$

By grouping terms we obtain

$$Q(t) = k_0 P_0|P_0|^{\frac{1}{n}-1} + \qquad \text{(CONT.)}$$

$$\rho\left[\frac{k_0}{n} P_1 |P_0|^{\frac{1}{n}-1} + k_1 |P_0|^{\frac{1}{n}-1}\frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1}\right)\right]$$

$$+ \rho^2\left[\frac{k_0}{n} P_2 |P_0|^{\frac{1}{n}-1} + \frac{k_0(1-n)P_1^2}{2n^2 \; P_0} |P_0|^{1/n-1}\right.$$

$$+ k_1 \frac{(1-n)}{n}\frac{P_1}{P_0} |P_0|^{\frac{1}{n}-1}\frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1}\right)$$

$$+ \frac{k_1}{n} |P_0|^{\frac{1}{n}-1}\frac{d}{dt}\left(P_1 |P_0|^{\frac{1}{n}-1}\right)$$

$$- k_2 (n-1)\frac{|P_0|^{\frac{1}{n}-1}}{P_0}\left(\frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1}\right)\right)^2$$

$$\left. + 2k_2 |P_0|^{\frac{1}{n}-1}\frac{d}{dt}\left(|P_0|^{\frac{1}{n}-1}\frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1}\right)\right)\right] \qquad (6.95)$$

But Q(t) is a fixed quantity and independent of the fluid inertia. Therefore, the coefficients of ρ and ρ^2 must vanish in equation (6.95) giving

$$Q(t) = k_0 P_0 |P_0|^{\frac{1}{n}-1} \qquad (6.96)$$

which, on rearrangement gives

$$P_0 = \frac{1}{k_0^n} Q|Q|^{n-1} \qquad (6.97)$$

Substituting equation (6.87) into equation (6.97) gives

$$P_0 = \frac{2K(3n+1)^n}{\pi^n n^n a^{3n+1}} Q|Q|^{n-1} \qquad (6.98)$$

By comparing terms of order ρ^2 in equation (6.95) we obtain

$$\frac{k_0}{n} P_1 |P_0|^{\frac{1}{n}-1} + k_1 |P_0|^{\frac{1}{n}-1} \frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1} \right) = 0 \qquad (6.99)$$

which, on rearrangement, gives

$$P_1 = -n \frac{k_1}{k_0} \frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1} \right) \qquad (6.100)$$

Using the expression for k_0 and k_1 given by equations (6.87) and (6.88) we have

$$P_1 = \frac{3n+1}{\pi a^2 (2n+1)} \frac{d}{dt} Q(t) \qquad (6.101)$$

By comparing terms of order ρ^2 in equation (6.95) and dividing through by $|P_0|^{-1+1/n}$ we obtain

$$k_0 \left[\frac{P_2}{n} + \frac{(1-n)}{2n^2} \frac{P_1^2}{P_0} \right] + k_1 \frac{(1-n)}{n} \frac{P_1}{P_0} \frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1} \right)$$

$$+ \frac{k_1}{n} \frac{d}{dt}\left(P_1 |P_0|^{\frac{1}{n}-1} \right) - k_2 \frac{(n-1)}{P_0} \left(\frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1} \right) \right)^2$$

$$+ 2 k_2 \frac{d}{dt}\left(|P_0|^{\frac{1}{n}-1} \frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1} \right) \right) = 0 \qquad (6.102)$$

Substituting P_1 from equation (6.100) into this equation and grouping terms gives

$$P_2 = \frac{n}{k_0} \left(k_1^2 - 2 k_0 k_2 \right) \left[|P_0|^{\frac{1}{n}-1} \frac{d^2}{dt^2}\left(P_0 |P_0|^{\frac{1}{n}-1} \right) \right.$$

$$+ \frac{1-n}{2 P_0} \left(\frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1} \right) \right)^2$$

$$\left. + \frac{d}{dt}\left(|P_0|^{\frac{1}{n}-1} \right) \frac{d}{dt}\left(P_0 |P_0|^{\frac{1}{n}-1} \right) \right] \qquad (6.103)$$

Substituting the expressions for k_0, k_1 and k_2 from equations (6.87), (6.88) and (6.89) gives

$$P_2 = \frac{-n^3 a^{2+\frac{2}{n}}}{(2k)^{2/n}(3n+1)(2n+1)(5n+3)}\Bigg[|P_0|^{\frac{1}{n}-1}\frac{d^2}{dt^2}\left(P_0|P_0|^{\frac{1}{n}-1}\right)$$

$$+\frac{(1-n)}{2P_0}\left(\frac{d}{dt}\left(P_0|P_0|^{-\frac{1}{n}-1}\right)\right)^2$$

$$+\frac{d}{dt}\left(|P_0|^{\frac{1}{n}-1}\right)\frac{d}{dt}\left(P_0|P_0|^{\frac{1}{n}-1}\right)\Bigg] \qquad (6.104)$$

Substituting equation (6.97) into equation (6.104) gives

$$P_2 = \frac{-n^{n+1}(3n+1)^{1-n}\pi^{n-2}a^{3n-3}}{2k(2n+1)^2(5n+3)\,Q|Q|^{n-1}}\Bigg[QQ'' + \frac{1-n}{2}Q'^2$$

$$+ Q|Q|^{n-1}\frac{d}{dt}\left(|Q|^{1-n}\right)Q'\Bigg] \qquad (6.105)$$

Equation (6.105) simplifies to give

$$P_2 = \frac{-n^{n+1}(3n+1)^{1-n}\pi^{n-2}a^{3n-3}}{2k(2n+1)^2(5n+3)}\left[\frac{QQ''+\frac{3}{2}(1-n)Q'^2}{Q|Q|^{n-1}}\right] \qquad (6.106)$$

Substituting the expression for P_0, P_1 and P_2 given in equations (6.97), (6.100) and (6.106) into equation (6.90) we obtain

$$P(t) = \frac{2k(3n+1)^n}{\pi^n n^n a^{3n+1}}Q|Q|^{n-1} + \frac{\rho(3n+1)}{\pi a^2(2n+1)}Q'$$

$$-\rho^2\frac{n^{n+1}(3n+1)^{1-n}\pi^{n-2}a^{3n-3}}{2k(2n+1)^2(5n+3)}\left[\frac{QQ''+\frac{3}{2}(1-n)Q'^2}{Q|Q|^{n-1}}\right] \qquad (6.107)$$

and $P(t)$ is now expressed in terms of $Q(t)$. Substituting equation (6.38) into equation (6.107) and using equation (6.41) it is evident that a singularity occurs in $P(t)$ when $Q(t) = 0$ and $\epsilon \geqslant 1$. The mean theoretical pressure gradient, $P(t)$ is, using equation (6.107) given by

$$\bar{P}(t) = \frac{2K(3n+1)^n}{\pi^n n^n a^{3n+1}} \langle Q|Q|^{n-1}\rangle - \frac{\rho^2 n^{n+1}(3n+1)^{1-n}\pi^{n-2}a^{3n-3}}{2K(2n+1)^2(5n+3)} \left\langle \frac{QQ'' + \frac{3}{2}(1-n)Q'^2}{Q|Q|^{n-1}} \right\rangle \tag{6.108}$$

or, using equation (6.90)

$$\bar{P}(t) = \langle P_0\rangle + \rho^2\langle P_2\rangle \tag{6.109}$$

6.3.3 Non-Dimensional Analysis

Let

$$\rho^2 P_2 = \bar{P}_s P_2^* \tag{6.110}$$

$$T = \frac{\omega t}{2\pi} \tag{6.111}$$

where $\bar{P}_s$ is the mean steady pressure gradient given by equation (6.12).

Using equations (6.106) and (6.38)

$$\rho^2 P_2 = \frac{n^{n+1}\rho^2(3n+1)^{1-n}\pi^{n-2}a^{3n-3}\pi^2 a^4 \bar{u}^2 \omega^2}{2K(2n+1)(5n+3)\pi^n a^{2n}\bar{u}^n Q^*|Q^*|^{n-1}4\pi^2}$$

$$\times\left[\frac{3}{2}(n-1)(Q^{*\prime})^2 - Q^* Q^{*\prime\prime}\right] \tag{6.112}$$

$$= \frac{\rho^2 n^{n+1}(3n+1)^{1-n}a^{n+1}\omega^2}{8k(2n+1)^2(5n+3)\pi^2\bar{u}^{n-2}}\left[\frac{\frac{3}{2}(n-1)(Q^{*\prime})^2 - Q^*Q^{*\prime\prime}}{Q^*|Q^*|^{n-1}}\right] \tag{6.113}$$

Dividing equation (6.113) through by P_s, using equations (6.20), (6.22), (6.24) and (6.110) and simplifying we obtain

$$P_2^* = \frac{F^2(3n+1)n}{256(2n+1)^2(5n+3)}\left[\frac{3/2(n-1)(Q^{*\prime})^2 - Q^*Q^{*\prime\prime}}{Q^*|Q^*|^{n-1}}\right] \tag{6.114}$$

Using equation (6.98) we obtain

$$P_o = 2K\left(\frac{3n+1}{n}\right)^n \frac{\bar{u}^n}{a^{n+1}} \; Q^*|Q^*|^{n-1} \tag{6.115}$$

Therefore

$$P_o^* = \frac{P_o}{\bar{P}_s} = Q^*|Q^*|^{n-1} \tag{6.116}$$

Finally the normalised pressure gradient is given by

$$P^* = \frac{P}{\bar{P}_s} \tag{6.117}$$

From equation (6.109) we have

$$\bar{P}^* = \langle P_o^* \rangle + \langle P_2^* \rangle \tag{6.118}$$

and

$$\bar{P}^* = \langle Q^*|Q^*|^{n-1}\rangle + \frac{F^2(3n+1)n}{256(2n+1)^2(5n+3)} \left\langle \frac{\frac{3}{2}(n-1)(Q^{*\prime})^2 - Q^*Q^{*\prime\prime}}{Q^*|Q^*|^{n-1}} \right\rangle \tag{6.119}$$

Therefore, the normalised mean pressure gradient is given by

$$\bar{P}^* = \int_0^1 Q^*|Q^*|^{n-1}\,dT + \frac{F^2 n(3n+1)}{256(2n+1)^2(5n+3)} \int_0^1 \frac{\frac{3}{2}(n-1)(Q^{*\prime})^2 - Q^*Q^{*\prime\prime}}{Q^*|Q^*|^{n-1}}\,dT \tag{6.120}$$

Since $Q^* = 1 + \epsilon \cos 2\pi T$ then it appears that a singularity occurs in the second integrand when $\epsilon \geqslant 1$. However, this is not the case since rewriting this expression we have

$$\int_0^1 \frac{\frac{3}{2}(n-1)(Q^{*\prime})^2}{Q^*|Q^*|^{n-1}}\,dT = \frac{3}{2}(n-1)\int_0^1 \left(\frac{Q^{*\prime}}{Q^*|Q^*|^{n-1}}\right) Q^{*\prime}\,dT \tag{6.121}$$

$$= \frac{3}{2}(n-1)\left[Q^{*'}|Q^*|^{1-n}\right]_0^1 - \frac{3}{2}(n-1)\int_0^1 \frac{|Q^*|^{1-n}}{1-n} Q^{*''} dT \qquad (6.122)$$

$$= \frac{3}{2}\int_0^1 |Q^*|^{1-n} Q^{*''} dT \qquad (6.123)$$

The second integral in (6.120) becomes

$$\int_0^1 \left(\frac{3(n-1)(Q^{*'})^2}{2Q^*|Q^*|^{n-1}} - \frac{Q^{*''}}{|Q^*|^{n-1}} \right) dT$$

$$= \int_0^1 \left(\frac{3}{2}|Q^*|^{1-n}Q^{*''} - Q^{*''}|Q^*|^{1-n} \right) dT \qquad (6.124)$$

$$= \frac{1}{2}\int_0^1 |Q^*|^{1-n} Q^{*''} dT \qquad (6.125)$$

and the normalised mean pressure gradient is given by

$$\bar{p}^* = \int_0^1 Q^*|Q^*|^{n-1} dT + \frac{F^2(3n+1)n}{512(2n+1)^2(5n+3)} \int_0^1 |Q^*|^{1-n} Q^{*''} dT \qquad (6.126)$$

The results from equation (6.126) are plotted in Figs. 13 - 16.

6.4 Results and Discussion

As expected there is total agreement in the Newtonian case, between the data derived numerically and that obtained using the perturbation theory. There is also good agreement between the two sets of data when the fluid inertia is zero (i.e. when F=0). This is evident from Figs. 13 - 16 and from Tables 4 - 7 and Table 12.

The numerical variation of the normalised mean pressure gradient against F for $\epsilon = 10.0$ and $\epsilon = 5.0$ is small. Therefore for this reason it is difficult to know whether there is good agreement between the perturbation analysis and the numerical data. However, for the case when $\epsilon = 1.0$ or 0.5 the variation of the mean pressure gradient with F is more pronounced and the disagreement between the two relevant curves is more noticeable. Since the range of validity of the perturbation theory and of the numerical procedure is uncertain then this disagreement might be removed by using a modified perturbation theory which takes into account higher order terms and a numerical procedure which is free of instabilities. It should be noted that, since terms of third order and higher were ignored in the perturbation theory then the variation of mean pressure gradient against F is quadratic. The modified perturbation theory mentioned might be obtained by including terms of third (and possibly fourth) order and ignoring all higher order terms.

The downwards direction of variation of the mean pressure gradient with F, predicted by the perturbation theory, is consistent with the corresponding numerically derived variation of these quantities. Also, trends similar to those evident in the complete, numerically derived curves, have been observed from the results of the experimental work carried out by Chakrabati and Davies already mentioned in the introduction to this Chapter. However, Davies[47] (private communication 1986) has recently shown that similar trends occur when elasticity is included in the theory.

Figures 17 - 20 show the relationship between phase angle and F for various ϵ and N. The variation of the phase angle with F shows the same trend with N for all ϵ, being more pronounced for larger ϵ. As F gets large then, in all cases, the phase angle tends to a 90° lag.

Figures 21 - 24 show the relationship between the fundamental pressure amplitude and F for various values of ϵ and N. The variation of the pressure amplitude with F shows the same trend with N for all ϵ. For large F the pressure amplitude, for each value of ϵ, appears to be independent of N and proportional to a power of F.

Finally, Fig. 25 shows the relationship between the normalised mean pressure gradient and ϵ for various values of F when N = 0.5. In general the flow enhancement increases with increasing and decreasing fluid inertia. Interestingly, however, when $\epsilon = 1$ then the flow enhancement is practically independent of the fluid inertia.

The work presented in this chapter has shown the effect of a realistic viscosity function and the effect of fluid inertia on the flow enhancement and pressure gradient reduction predictions. It is hoped that this work will be extended to include the effect of elasticity. However, it is important to choose a model which gives good predictions for steady shear behaviour and oscillatory shear flow behaviour.

In the oil drilling industry, large positive displacement mechanical pumps are used to circulate the drilling fluid. These pumps use an oscillatory mechanism which results in a pulsatile pressure gradient imposed on the fluid. The properties of the fluid will obviously be affected by the superposition of an oscillation on the steady flow. This type of flow behaviour also occurs in the removal of rock cuttings from the recovered oil drilling fluid. Due to the cost of this fluid it is necessary to separate the drill cuttings from it enabling re-circulation to take place. This is achieved by passing the contaminated oil drilling fluid over a vibrating table. The work presented in this Chapter will obviously be relevant in describing the pulsatile flow behaviour of fluids used in the oil drilling industry.

APPENDIX 1

THE GOVERNING EQUATION FOR A SPHERE FALLING SLOWLY THROUGH A NEWTONIAN FLUID

The constituitive equations for a Newtonian fluid are, using spherical polar coordinates and taking the symmetry of Fig. 3 into account:

$$\tau_{rr} = -2\eta \frac{\partial V_r}{\partial r} \tag{A1.1}$$

$$\tau_{\theta\theta} = -2\eta\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right) \tag{A1.2}$$

$$\tau_{\phi\phi} = -2\eta\left(\frac{V_r}{r} + V_\theta \frac{\cot\theta}{r}\right) \tag{A1.3}$$

$$\tau_{r\theta} = -\eta\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right) \tag{A1.4}$$

The equations of motion, ignoring the inertial terms are:

$$-\frac{\partial p}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \tau_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{r\theta}\sin\theta\right) - \frac{(\tau_{\theta\theta} + \tau_{\phi\phi})}{r} \tag{A1.5}$$

and

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 \tau_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{\theta\theta}\sin\theta\right) - \frac{\tau_{\phi\phi}\cot\theta}{r} \tag{A1.6}$$

The equation of continuity is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(V_\theta\sin\theta\right)=0 \qquad (A1.7)$$

From the Stoke's streamline formulation

$$V_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\quad,\qquad V_\theta=\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \qquad (A1.8)$$

From the equation of continuity

$$\frac{\partial V_r}{\partial r}+\frac{2}{r}V_r+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{\cot\theta}{r}V_\theta=0 \qquad (A1.9)$$

Using

$$\tau_{\theta\theta}+\tau_{\phi\phi}=-2\eta\left[\frac{V_r}{r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{V_r}{r}+V_\theta\frac{\cot\theta}{r}\right] \qquad (A1.10)$$

$$= 2\eta\frac{\partial V_r}{\partial r} \qquad (A1.11)$$

the equations of motion become

$$\frac{1}{\eta}\frac{\partial p}{\partial r}=\frac{2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V_r}{\partial r}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)+\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\right]+\frac{2}{r}\frac{\partial V_r}{\partial r} \qquad (A1.12)$$

i.e.

$$\frac{1}{\eta}\frac{\partial p}{\partial r}=\frac{2}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial V_r}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)+\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\right] \qquad (A1.13)$$

and

$$\frac{1}{\eta}\frac{\partial p}{\partial\theta}=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)+\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\right]+\frac{2}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{V_r}{r}\right)\right]-2\cot\theta\left[\frac{V_r}{r}+V_\theta\frac{\cot\theta}{r}\right] \qquad (A1.14)$$

Written in terms of ψ the equations of motion become

$$\frac{1}{\eta}\frac{\partial p}{\partial r} = -\frac{2}{r^3\sin\theta}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right)\right)$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right)\right)$$

$$-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) \quad (A1.15)$$

$$=\frac{2}{r^3\sin\theta}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right)\right)+\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right)$$

$$-\frac{1}{r^4\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) \quad (A1.16)$$

and

$$\frac{1}{\eta}\frac{\partial p}{\partial\theta} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial r}\right)\right)$$

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right)$$

$$+\frac{2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)\right)$$

$$+\frac{2\cot\theta}{r^3\sin\theta}\frac{\partial\psi}{\partial\theta}-\frac{2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r^3}\frac{\partial\psi}{\partial\theta}\right)$$

$$-\frac{2\cot^2\theta}{r^2\sin\theta}\cdot\frac{\partial\psi}{\partial r} \quad (A1.17)$$

$$=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right)\right)-\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$$

$$+\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right)\right)$$

$$+\frac{2\cot\theta}{r^3\sin\theta}\frac{\partial\psi}{\partial\theta}-\frac{2}{r^3\sin\theta}\frac{\partial^2\psi}{\partial\theta^2}-\frac{2\cot^2\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial r}$$

$$(A1.18)$$

i.e.

$$\frac{1}{\eta}\frac{\partial p}{\partial\theta} = \frac{1}{r^2\sin\theta}\left(r^2\frac{\partial^3\psi}{\partial r^3} - 2\frac{\partial\psi}{\partial r}\right)$$

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$$

$$+\frac{2}{r^2\sin\theta}\left(\frac{\partial^3\psi}{\partial\theta^2\partial r} - \cot\theta\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\psi}{\partial r}\right)$$

$$+\frac{2\cot\theta}{r^3\sin\theta}\frac{\partial\psi}{\partial\theta} - \frac{2}{r^3\sin\theta}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2\cot^2\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial r} \qquad (A1.19)$$

$$= \frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^3} + \frac{2}{r^2}\frac{\partial\psi}{\partial r}\left(\frac{-1}{\sin\theta} + \frac{1}{\sin^3\theta} - \frac{\cot^2\theta}{\sin\theta}\right)$$

$$-\frac{1}{r^2\sin\theta}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{1}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{2}{r^2\sin\theta}\frac{\partial^3\psi}{\partial\theta^2\partial r}$$

$$-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{2\cot\theta}{r^3\sin\theta}\frac{\partial\psi}{\partial\theta} - \frac{2}{r^3\sin\theta}\frac{\partial^2\psi}{\partial\theta^2} \qquad (A1.20)$$

i.e.

$$\frac{1}{\eta}\frac{\partial p}{\partial\theta} = \frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^3} + \left(\frac{1}{r^2\sin\theta}\frac{\partial^3\psi}{\partial\theta^2\partial r} - \frac{\cot\theta}{r^2\sin\theta}\frac{\partial^2\psi}{\partial r\partial\theta}\right)$$

$$-\frac{2}{r^3\sin\theta}\frac{\partial^2\psi}{\partial\theta^2} + \frac{2\cot\theta}{r^3\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad (A1.21)$$

$$= \frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) - \frac{2}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) \qquad (A1.22)$$

Therefore

$$\frac{1}{\eta}\frac{\partial p}{\partial\theta} = \frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^3} + \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) \qquad (A1.23)$$

Now

$$\frac{1}{\eta}\frac{\partial p}{\partial r} = \frac{-2}{r^3 \sin\theta}\frac{\partial}{\partial r}\left(r\frac{\partial^2 \psi}{\partial r \partial\theta} - 2\frac{\partial\psi}{\partial\theta}\right)$$

$$+\frac{1}{\sin\theta}\left(-\frac{2}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta\partial r^2}\right)$$

$$-\frac{1}{r^4\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) \quad \text{(A1.24)}$$

$$= \frac{-2}{r^2\sin\theta}\frac{\partial^3\psi}{\partial r^2\partial\theta} + \frac{1}{r^2\sin\theta}\frac{\partial^3\psi}{\partial\theta\partial r^2}$$

$$-\frac{1}{r^4\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) \quad \text{(A1.25)}$$

Therefore

$$\frac{1}{\eta}\frac{\partial p}{\partial r} = -\frac{1}{r^2\sin\theta}\frac{\partial^3\psi}{\partial r^2\partial\theta} - \frac{1}{r^4\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) \quad \text{(A1.26)}$$

Eliminating p from equations (A1.23) and (A1.26) by setting $\frac{\partial^2 p}{\partial r\partial\theta} = \frac{\partial^2 p}{\partial\theta\partial r}$

$$\frac{1}{\sin\theta}\frac{\partial^4\psi}{\partial r^4} + \frac{\partial^2}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^2\partial\theta}\right) - \frac{1}{r^4}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right)\right)$$

i.e. (A1.27)

$$0 = \frac{\partial^4\psi}{\partial r^4} + \sin\theta\frac{\partial^2}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right)$$

$$+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial^3\psi}{\partial r^2\partial\theta}\right)$$

$$+\frac{\sin\theta}{r^4}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right)\right)$$

(A1.28)

Therefore

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right)\left(\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right) = 0 \qquad (A1.29)$$

i.e.

$$E^4\psi = 0 \qquad (A1.30)$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \qquad (A1.31)$$

Equation (A1.31) is the governing equation for the Newtonian fluid.

APPENDIX 2

DERIVING THE BOUNDARY CONDITIONS FOR ψ_{NN} WHEN THE OUTER SPHERE HAS A FINITE RADIUS

The boundary conditions derived here first are true for both Newtonian and power law fluids. The stream function, ψ, refers therefore to either fluid.

Let the inner sphere have a radius a and the outer sphere a radius b. Using equations (4.3) and (4.4) and the no slip condition on the inner sphere, we obtain

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = \left.\frac{\partial\psi}{\partial\theta}\right|_{r=a} = 0 \qquad (A2.1)$$

and

$$\psi\Big|_{r=a} = \text{Const} \qquad (A2.2)$$

The constant is arbitrary and without loss of generality is chosen to be zero. Therefore

$$\psi\Big|_{r=a} = 0 \qquad (A2.3)$$

From Fig. 3 and the no slip condition on the outer sphere, we obtain

$$V_\theta\Big|_{r=b} = -U\sin\theta \qquad (A2.4)$$

and

$$V_r\Big|_{r=b} = U\cos\theta \qquad (A2.5)$$

From equations (4.3), (4.4), (A2.4) and (A2.5), we obtain

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{r=b} = -Ub^2\sin\theta\cos\theta \qquad (A2.6)$$

and

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=b} = -Ub\sin^2\theta \qquad (A2.7)$$

Equation (A2.6) can be rewritten as

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{r=b} = -\frac{Ub^2}{2}\frac{\partial}{\partial\theta}\sin^2\theta \qquad (A2.8)$$

and

$$\psi\Big|_{r=b} = -\frac{Ub^2}{2}\sin^2\theta + \text{const} \qquad (A2.9)$$

From equation (4.4), we obtain

$$\left.\frac{\partial\psi}{\partial r}\right|_{\theta=0} = \left.\frac{\partial\psi}{\partial r}\right|_{r=\pi} = 0 \qquad (A2.10)$$

From equations (A2.3) and (A2.10)

$$\psi\Big|_{\theta=0} = \psi\Big|_{\theta=\pi} = 0 \qquad (A2.11)$$

From equations (A2.9) and (A2.11) we hve

$$\psi\Big|_{r=b} = -\frac{1}{2}Ub^2\sin^2\theta \qquad (A2.12)$$

Let ψ now refer to the stream function for the power law fluid. Let the Newtonian stream function be ψ_N. Thus

$$\psi_N\Big|_{r=a} = 0$$

$$\psi_N\Big|_{r=b} = -\frac{1}{2}Ub^2\sin^2\theta$$

$$\left.\frac{\partial\psi_N}{\partial r}\right|_{r=a} = 0 \qquad (A2.13)$$

$$\left.\frac{\partial\psi_N}{\partial r}\right|_{r=b} = -Ub\sin^2\theta$$

$$\left.\frac{\partial\psi_N}{\partial r}\right|_{\theta=0} = \left.\frac{\partial\psi_N}{\partial r}\right|_{\theta=\pi} = 0$$

Using equation (4.26) we obtain

$$\psi_{NN}\Big|_{r=a} = \psi_{NN}\Big|_{r=b} = 0$$

$$\frac{\partial\psi_{NN}}{\partial r}\Big|_{r=a} = \frac{\partial\psi_{NN}}{\partial r}\Big|_{r=b} = 0 \qquad \text{(A2.14)}$$

and

$$\frac{\partial\psi_{NN}}{\partial\theta}\Big|_{\theta=0} = \frac{\partial\psi_{NN}}{\partial\theta}\Big|_{\theta=\pi} = 0$$

the required boundary conditions for ψ_{NN} when the outer sphere has a finite radius.

APPENDIX 3

DERIVING THE NEWTONIAN STREAM FUNCTION WHEN THE OUTER SPHERE HAS A FINITE OR AN INFINITE RADIUS

The Newtonian stream function, ψ_N, was obtained by solving equation (A1.31) under the boundary conditions for ψ_N derived in Appendix 2. A variable separable solution of the form

$$\psi_N = F(r)\sin^2\theta \qquad (A3.1)$$

was assumed. Substituting equation (A3.1) into equation (A1.29) we obtain

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)F(r) = 0 \qquad (A3.2)$$

Let

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)F(r) = F^*(r) \qquad (A3.3)$$

Then equation (A3.2) may be rewritten as

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)F^*(r) = 0 \qquad (A3.4)$$

By inspection, a solution is given by

$$F^*(r) = \frac{a}{r} + cr^2 \qquad (A3.5)$$

where a and c are arbitrary constants. From equations (A3.3) and (A3.5) we have

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)F(r) = \frac{a}{r} + cr^2 \qquad (A3.6)$$

By inspection, a particular integral is given by

$$F_p(r) = b^* r + d^* r^4 \qquad \text{(A3.7)}$$

and the complementary function is given by

$$F_c(r) = \frac{a^*}{r} + c^* r^2 \qquad \text{(A3.8)}$$

where a*, b*, c* and d* are arbitrary constants. Let

$$a^* = -\frac{U}{2} a^3 A$$

$$b^* = -\frac{U}{2} a B \qquad \text{(A3.9)}$$

and

$$c^* = -\frac{U}{2} C$$

$$d^* = -\frac{U}{2} \frac{D}{a^2}$$

Then

$$\psi = (F_p(r) + F_c(r)) \sin^2\theta \qquad \text{(A3.10)}$$

i.e.

$$\psi = -\frac{1}{2} U a^2 \left(A\frac{a}{r} + B\frac{r}{a} + C\frac{r^2}{a^2} + D\frac{r^4}{a^4} \right) \sin^2\theta \qquad \text{(A3.11)}$$

The constants A, B, C and D are now derived. From equations (A2.1), (A2.3), (A2.7), (A2.12) and (A3.11) we obtain

$$A + B + C + D = 0 \qquad \text{(A3.12)}$$

$$-A + B + 2C + 4D = 0 \qquad \text{(A3.13)}$$

$$A + Bd^2 + Cd^3 + Dd^5 = d^3 \qquad \text{(A3.14)}$$

and

$$-A + Bd^2 + 2Cd^3 + 4Dd^5 = 2d^3 \qquad \text{(A3.15)}$$

Eliminating A, we have

$$2B + 3C + 5D = 0 \qquad \text{(A3.16)}$$

from (A3.12) + (A3.13)

$$2B + 3Cd + 5Dd^3 = 3d \qquad (A3.17)$$

from (A3.14) + (A3.15)

$$3C(d-1) + 5D(d^3-1) = 3d \qquad (A3.18)$$

from (A3.17) - (A3.16)

$$(d^2-1)B + (d^3-1)C + (d^5-1)D = d^3 \qquad (A3.19)$$

from (A3.12) - (A3.14) and

$$(d^2-1)B + (d^3-1)2C + 4(d^5-1)D = 2d^3 \qquad (A3.20)$$

from (A3.13) - (A3.15).

Subtracting equation (A3.19) from (A3.20) we obtain

$$(d^3-1)C + 3(d^5-1)D = d^3 \qquad (A3.21)$$

Therefore

$$C = \frac{d^3 - 3(d^5-1)D}{d^3-1} \qquad (A3.22)$$

Substituting equation (A3.22) into equation (A3.18) we obtan

$$3\left[\frac{d^3 - 3(d^5-1)D}{1+d+d^2}\right] + 5D(d^3-1) = 3d \qquad (A3.23)$$

But

$$d^3-1 = (d-1)(d^2+d+1) \qquad (A3.24)$$

and

$$d^5-1 = (d-1)(d^4+d^3+d^2+1) \qquad (A3.25)$$

Transposing equation (A3.23) we obtain

$$D\left[d + 6d^2 + d^3 - 4d^4 - 4\right] = \frac{-3d(1+d)}{(d-1)} \qquad (A3.26)$$

which may be written as

$$D(d-1)^2(4d^2+7d+4) = \frac{-3d(1+d)}{d-1} \quad (A3.27)$$

and

$$D = \frac{-3d(1+d)}{(d-1)^3(4d^2+7d+4)} \quad (A3.28)$$

By back substitution into equations (A3.18), (A3.16) and (A3.12) we obtain

$$C = \frac{d(9+9d+4d^2+4d^3+4d^4)}{(d-1)^3(4+7d+4d^2)} \quad (A3.29)$$

$$B = \frac{-6d(1+ \text{[illegible]})}{(d-1)^3(4+7d+4d^2)} \quad (A3.30)$$

and

$$A = \frac{2d^3(1+d+d^2)}{(d-1)^3(4+7d+4d^2)} \quad (A3.31)$$

and the expression for ψ_N is known.

For an unbounded Newtonian fluid $d \to \infty$ and we obtain

$$A = \frac{1}{2} \quad (A3.32)$$

$$B = -\frac{3}{2} \quad (A3.33)$$

$$C = 1 \quad (A3.34)$$

$$D = 0 \quad (A3.35)$$

APPENDIX 4

EVALUATING THE INTEGRAL I_2

The integral, I_2, is given by

$$I_2 = \int_0^{\pi} \sin^3\theta \log \sin^2\theta \, d\theta \qquad (A4.1)$$

Now

$$d\cos\theta = -\sin\theta \, d\theta \qquad (A4.2)$$

Therefore

$$I_2 = \int_{\theta=\pi}^{0} (1-\cos^2\theta)\log(1-\cos^2\theta)\, d\cos\theta \qquad (A4.3)$$

Let

$$\cos\theta = x \qquad (A4.4)$$

Then

$$I_2 = \int_{-1}^{1} (1-x^2)\log(1-x^2)\, dx \qquad (A4.5)$$

Integrating by parts

$$I_2 = \left[\left(x - \frac{x^3}{3}\right) \log(1-x^2) \right]_{-1}^{1} + I \qquad (A4.6)$$

where

$$I = \int_{-1}^{1} \left(x - \frac{x^3}{3}\right) \frac{2x}{1-x^2}\, dx \qquad (A4.7)$$

$$= \frac{4}{3} \int_0^1 \frac{x^2(3-x^2)}{1-x^2}\, dx \qquad (A4.8)$$

By repeatedly expressing the numerator of the integrand in terms of the denominator we find that

$$I = -\frac{20}{9} + \frac{4}{3}\left[\log 2 - \log(1-x)\Big|_{x=1}\right] \qquad (A4.9)$$

Now

$$\left[\left(x - \frac{x^3}{3}\right)\log\left(1-x^2\right)\right]_{-1}^{1} = \frac{2}{3}\log(1-x^2)\Big|_{x=1} + \frac{2}{3}\log\left(1-x^2\right)\Big|_{x=-1} \qquad (A4.10)$$

$$= \frac{2}{3}\log(1+x)_{x=1} + \frac{2}{3}\log(1-x)_{x=1} + \frac{2}{3}\log(1+x)_{x=-1} + \frac{2}{3}\log(1-x)_{x=-1} \qquad (A4.11)$$

$$= \frac{4}{3}\log 2 + \frac{4}{3}\log(1-x)_{x=1} \qquad (A4.12)$$

Therefore

$$I_2 = -\frac{20}{9} + \frac{8}{3}\log 2 \qquad (A4.13)$$

R E F E R E N C E S

1. R.B. Bird, Dynamics of Polymeric Liquids, Vol. 2 (Kinetic Theory), J. Wiley and Sons, 1977

2. T.J. Chung, Finite Element Analysis in Fluid Dynamics, McGraw Hill, 1978

3. J.G. Oldroyd, Proc. Roy. Soc. Ldn., A200, 523, 1950

4. H. Henky, Ztschr. f. angew. Math. und Mech., Band 5, Heft 2, 144, 1925

5. L. Boltzmann, Ann. Phys. Chem. (Poggendorff), Erganzunsband 7, 624, 1876

6. J.G. Oldroyd, Proc. Roy. Soc. Ldn., A245, 278, 1958

7. R.S. Rivlin and J.L. Ericksen, J. Rat. Mech. Anal., 4, 323, 1955

8. A.E. Green and R.S. Rivlin, Arch. Rat. Mech. Anal., 1, 1, 1957

9. A.E. Green and R.S. Rivlin, Arch. Rat. Mech. Anal., 4, 387, 1960

10. B.D. Coleman and W. Noll, Arch. Rat. Mech. Anal., 6, 355, 1960

11. B.D. Coleman and W. Noll, Annal. N.Y. Acad. Sci., 89, 672, 1961

12. B.D. Coleman and W. Noll, Rev. Mod. Physics, 33, 239, 1961

13. K. Walters, Quart. J. Mech. Applied Math, 8, 444, 1960

14. R.B. Bird, Dynamics of Polymeric Liquids, Vol. 1 (Fluid Mechanics), J. Wiley and Sons, 1977

15. A.S Lodge, Elastic Liquids, N.Y. Acad. Press, 1966

16. K. Walters, Z.A.M.P., 21, 592, 1970

17. G.V. Chilingarian and P. Vorabutr, Drilling and Drilling Fluids, Elsevier, 1981

18. R. Lauzon and K. Reid, Oil and Gas Journal, 51, 1979

19. M. Zamora and R. Bleier, Imco Services, 1976

20. R.E. Robertson and H.E. Stiff, Soc. Pet. Eng. J., 16, 31, 1976

21. D.W. McEachern, A.I.Ch.E.J., 12, 328, 1966

22. M. Slawomirski, Int. J. Rock Mech. Min. Sci. Geomech. Abstr., 12, 115, 1975

23. P. Reynolds, Private Communication, 1982

24. A. Acharya, R.A. Mashelkar and J. Ulbrecht, Rheol. Acta, 15, 454, 1976

25. M.L. Wassermann and J.C. Slattery, A.I.Ch.E.J., 10, 383, 1964

26. V. Mohan and D. Venkateshwarlu, Int. J. Multiphase Flow, 2, 563, 1976

27. N. Yoshioka and K. Adachi, J. Chem. Eng., Japan, 6, 134, 1973

28. Y.I. Cho and J.P. Hartnett, J.N.N.F.M., 12, 243, 1983

29. R.P. Chhabra, J.N.N.F.M., 13, 225, 1983

30. M.J. Crochet, A.R. Davies and K. Walters, Numerical Simulation of Non-Newtonian Flow, Rheology Series 1, Elsevier, 1984

31. Gu Dazhi and R.I. Tanner, J.N.N.F.M., **17**, 1, 1985

32. G. Tiefenbruck and L.G. Leal, J.N.N.F.M., **10**, 115, 1982

33. M.B. Bush and N. Phan-Thien, J.N.N.F.M., **16**, 303, 1984

34. G. Strang and G.J. Fix, An Analysis of the Finite Element Method, Prentice-Hall, 1973

35. A.J. Davies, The Finite Element Method, Oxford Applied Math. and Comp. Sci. Series, 1980

36. R.P. Chhabra, C. Tiu, P.H.T. Uhlherr, Proc. 8th Int. Cong. Rheol., 2, 9, 1980

37. M.A. Lockyer, J.M. Davies and T.E.R. Jones, Proc. 8th Int. Cong. Rheol., 2, 127, 1980

38. H.A. Barnes, P. Townsend and K. Walters, Rheol. Acta, **10**, 517, 1971

39. M.T. Edwards, D.A. Nellist and W.L. Wilkinson, Ch. Eng. Sci., 27, 295, 1972

40. N. Phan-Thien and J. Dudek, J.N.N.F.M., **11**, 147, 1982

41. K. Walters and P. Townsend, Proc. 4th Int. Cong. Rheol., 4, 471, 1968

42. J.M. Davies, S. Bhumiratana and R.B. Bird, J.N.N.F.M., 3, 237, 1977/1978

43. N. Phan-Thien, J.N.N.F.M., 4, 167, 1978

44. N. Phan-Thien, Rheol. Acta, 19, 285, 1980

45. J.M. Davies and A. Chakrabati, Proc. 4th Int. Cong. Rheol., 2, 217, 1980

46. P. Townsend, Rheol. Acta, 12, 13, 1973

47. J.M. Davies, Private Communication, 1986

48. J.C. Slattery, A.I.Ch.E.J., 8, 663, 1962

49. Y. Tomita, Bull J.S.M.E., 2, 469, 1959

50. A. Fararouri and R.C. Kintner, Trans. Soc. Rheol., 5, 369, 1961

NOMENCLATURE

a	:	the radius of the fixed inner sphere.
	:	the radius of the circular pipe.
c_i	:	the coefficient of ϕ_i in the expansion for $\hat{u}$ as a linear combination of shape functions.
e	:	element e of the finite element mesh.
$e_{ik}^{(1)}$	:	rate of strain tensor.
g_{ik}	:	the metric tensor of a fixed coordinate system x^i.
$h(t-t')$	:	the influence function used by Coleman and Noll.
k_0, k_1, k_2	:	functions of a, K and n.
l	:	load vector.
$l^{(e)}$	:	element load vector.
n	:	power law index of a power law fluid
	:	material constant of a Herschel-Bulkley fluid.
p	:	an arbitrary isotropic pressure.
p_{ik}	:	Cauchy stress tensor.
p'_{ik}	:	extra stress tensor.
$-p_s$	:	steady pressure gradient.

r	:	radial distance.
	:	residual, $A\hat{u} - Au$.
u	:	the solution to a differential equation.
$\hat{u}$	:	a trial function, an approximation to u.
$\hat{u}^{(e)}$	:	local approximation to u in element e.
$\hat{u}_i^{(e)}$	:	local approximation to u at the ith local node of element e.
(u, v)	:	local coordinates.
$\bar{u}$	:	mean fluid velocity.
v	:	fluid velocity vector.
$v_{i,k}$	:	covariant derivative of v_i.
w	:	z component of fluid velocity vector.
w_0, w_1, w_2	:	coefficients of ρ^0, ρ^1 and ρ^2 in the perturbation expansion of w about $\rho = 0$.
w_i	:	weighting function.
A	:	material constant for the Robertson-Stiff fluid.
	:	a differential operator.
	:	coefficient in the equation for ψ_N.
A'	:	coefficient in the equation for ψ_{NN}.
A^*	:	coefficient in the final equation for ψ_{NN}.

A_1	:	a function of $\dot{\gamma}_{ik}$, r and θ.
A(t)	:	a function arising from integration with respect to r.
B	:	coefficient in the equation for ψ_N.
	:	a material constant for the Robertson-Stiff fluid.
B'	:	coefficient in the equation for ψ_{NN}.
B*	:	coefficient in the final equation for ψ_{NN}.
B_1	:	a function of $\dot{\gamma}_{ik}$, r and θ.
B(t)	:	a function arising from integration with respect to r.
C	:	a material constant for the Robertson-Stiff fluid.
	:	a material constant for the Casson fluid.
	:	coefficient in the equation for ψ_N.
C_{ik}	:	the Cauchy deformation tensor.
C'	:	coefficient in the equation for ψ_{NN}.
C_1	:	a function of $\dot{\gamma}_{ik}$, r and θ.
C_1^*	:	an expression used in the analytical expression for $F(\psi_N)$.
C(R)	:	the correction term applied to $V(R)_{ORIG}$.
D	:	coefficient in the equation for ψ_N.
D'	:	coefficient in the equation for ψ_{NN}.

D_1	:	a function of $\dot{\gamma}_{ik}$, r and θ.
D_1^*	:	an expression used in the analytical expression for $F(\psi_N)$.
E^2	:	the differential operator of the governing equation for a Newtonian or a power law fluid.
F	:	a non-dimensional variable relating ϖ, Re and $\bar{u}$.
F_D	:	the drag force on the inner sphere.
F(n)	:	the drag correction factor for a sphere falling slowly through an infinite power law fluid.
F(n,d)	:	the drag correction factor for a sphere falling slowly through a finite power law fluid.
$F(\psi_N)$	:	a function of ψ_N.
$\underset{-\infty}{\overset{t}{F}}[C_{ik}]$	:	a functional of C_{ik}.
$G_1, \ldots, G_8$	:	expressions used in the analytical expression for $F(\psi_N)$.
$G_i(e)$, $G_j(e)$	:	global node numbers corresponding to local node numbers i and j in element (e).
I_2	:	an integral which must be determined numerically.
K	:	consistency factor for the power law fluid.
	:	material constant for the Herschel-Bulkley fluid.

$L\psi_{NN}$	:	load vector for ψ_{NN}.
$L\omega_{NN}$	:	load vector for ω_{NN}.
$N(\tau)$	:	relaxation spectrum over τ.
$P_1, \ldots, P_5$	:	expressions used in the analyical expression for $F(\psi_N)$.
$P(T)$	:	pressure gradient.
$P^*(T)$	:	the T dependent factor of $P(T)$
P_0, P_1, P_2	:	coefficients of ρ^0, ρ^1, ρ^2 in the perturbation expansion of P about ρ equals zero.
$\bar{Q}$	:	the mean flow rate of the fluid.
Q^*	:	the flow rate non-dimensionalised with respect to the mean flow rate.
Q_{NUM}	:	numerical value of Q^* before correction.
Q_0, Q_1, Q_2	:	coefficients of ρ^0, ρ^1 and ρ^2 in the perturbation expansion of Q about ρ equals zero.
R	:	non-dimensional variable.
Re	:	Reynold's number.
S	:	stiffness matrix.
$S^{(e)}$	:	element stiffness matrix.
$S_{ij}^{(e)}$	:	i, jth member of $S^{(e)}$.

$S_1, ..., S_6$: expressions used in the analytical expression for $F(\psi_N)$.

T : non-dimensional variable.

T_{ij}, T^{ij}, T_i^j : covariant, contravariant and mixed tensors.

U : instantaneous velocity of the outer sphere.

$(U^{(e)}, V^{(e)})$: global coordinates of a point in element (e).

$(U_i^{(e)}, V_i^{(e)})$: global coordinates of the ith local node of elemet (e).

V : non-dimensional variable.

$V(R)_{ORIG}$: numerical value of V(R) before correction.

$V(R)_{NEW}$: numerical value of V(R) after correction.

V_{rNN} : the non-Newtonian radial velocity component.

V_r, V_θ, V_ϕ : components of the fluid velocity vector in spherial polar coordinates.

V_r, V_θ, V_z : components of the fluid velocity vector in cylindrical polar coordinates.

V_0, V_1, V_2 : the coefficients of ρ^0, ρ^1 and ρ^2 in the perturbation expansion for V about ρ equals zero.

X : variable of transformation.

Z : variable of transformation.

α	:	a function of the power law index of the power law fluid.
	:	material constant for an Ellis fluid.
α_1, α_5	:	material constants for the third order Rivlin-Ericksen fluids.
α_2, α_3	:	material constants for the third order Rivlin-Ericksen fluid.
	:	the second and third derivatives of ψ_{NN} with respect to r when r=a.
β_1	:	material constant for the third order Rivlin-Ericksen fluid.
β_2	:	material constant for the third order Rivlin-Ericksen fluid.
	:	second derivative of ω_{NN} with respect to r when r=a.
β_3	:	third derivative of ψ_{NN} with respect to r when r=a.
$\dot{\gamma}$	:	fluid shear rate.
δ_{ik}	:	unit tensor.
ϵ	:	ratio between the pulsed flow rate amplitude and the steady flow rate amplitude.
η	:	the fluid viscosity.
η_0	:	Newtonian viscosity.
	:	material constant for the Ellis fluid.
	:	material constant for the Oldroyd four constant fluid.
λ	:	a constant in the assumed linear relation of C(R) with R.

λ_1, λ_2 : material constants for the Oldroyd four constant fluid.

μ_o : viscosity at infinite shear rate for a Bingham fluid.

: material constant for the Oldroyd four constant fluid.

π_{ik} : total stress tensor.

ρ : fluid density.

τ : the relaxation time.

: the magnitude of the extra stress tensor.

τ_{ij} : the extra stress tensor.

τ_o : material constant for a Herschel-Bulkley fluid.

: yield stress for a Bingham fluid.

$\tau_{1/2}$: material constant for an Ellis fluid.

ϕ_i : a shape (basis) function.

$\phi_1^{(e)},\ldots,\phi_4^{(e)}$: local shape functions for element (e).

ψ : stream function.

ψ_N : Newtonian stream function.

ψ_{NN} : non-Newtonian contribution to the stream function.

$\underline{\psi}_{NN}$: coloumn vector of ψ_{NN} values at node points of finite element mesh.

ω	:	the frequency of the imposed pulsatile pressure gradient (rads s^{-1}).
	:	equal to $E^2\psi$.
ω_N	:	equal to $E^2\psi_N$.
ω_{NN}	:	equal to $E^2\psi_{NN}$.
$\underline{\omega}_{NN}$	:	column vector of values of ω_{NN} at the nodes of the finite element mesh.
Γ	:	the boundary of Ω
Ω	:	a physical region in the (r, θ) plane.
D/Dt	:	material derivative.
$\vartheta/\vartheta t$	:	a derivative independent of absolute motion in space.
$-\partial p/\partial Z$	:	pressure gradient in tube.
$\partial\psi_{NN}/\partial n$	:	normal derivative of ψ_{NN}.
$\partial\omega_{NN}/\partial n$	:	normal derivative of ω_{NN}.
I, II, III	:	the invariants of the rate of strain tensor.
$\langle . \rangle$	:	average value taken over one cycle.
$\langle r,w_i \rangle$	:	the inner product of r with w_i.

FIGURE 1

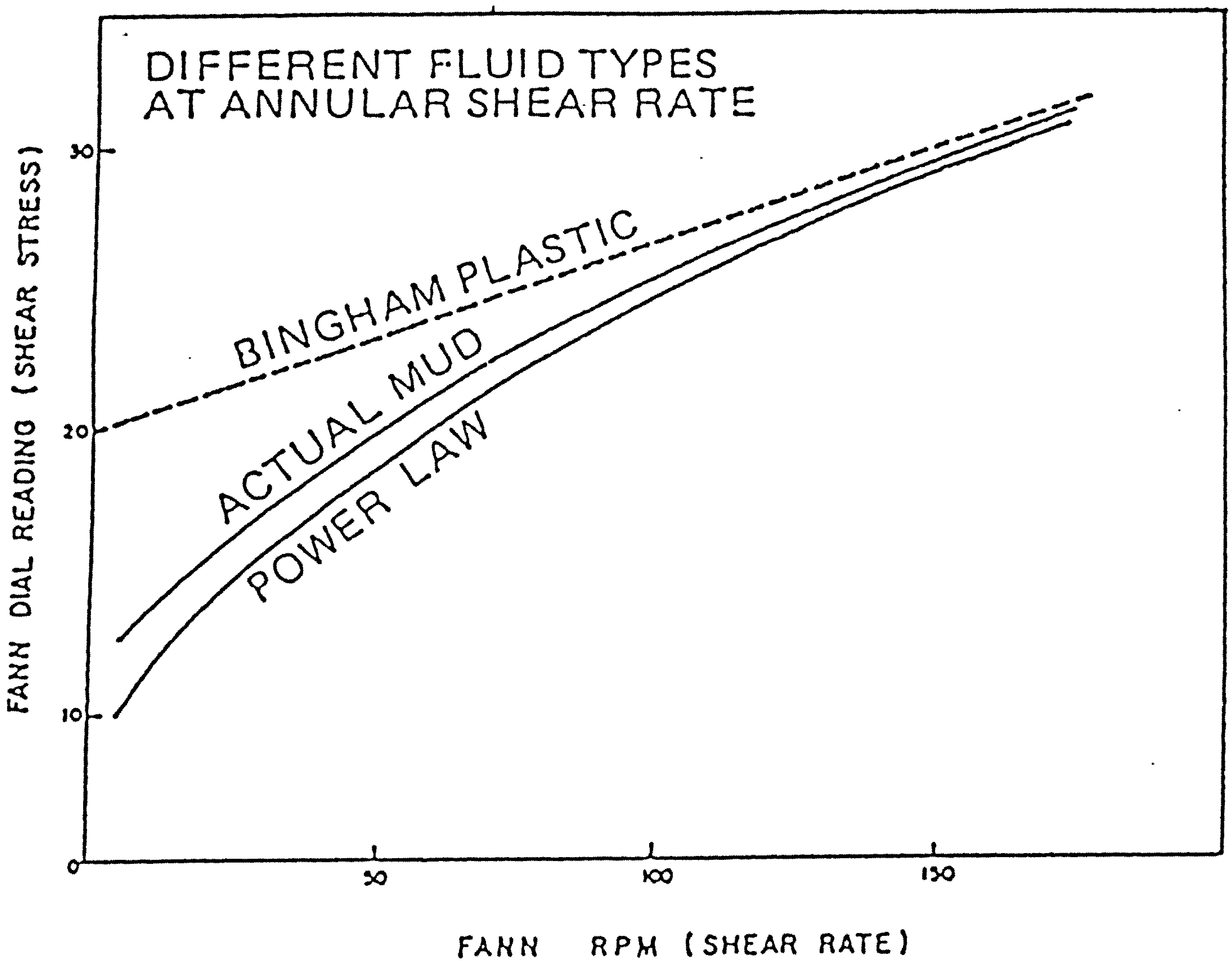
THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN SHEAR STRESS AND SHEAR RATE FOR A COMMERCIAL DRILLING FLUID.

THE SAME RELATIONSHIP IS ALSO SHOWN FOR

(i) THE BINGHAM FLUID

AND (ii) THE POWER LAW FLUID

CONSTRUCTED TO REPRESENT THE DRILLING FLUID.


DIFFERENT FLUID TYPES
AT ANNULAR SHEAR RATE
BINGHAM PLASTIC
ACTUAL MUD
POWER LAW
FANN DIAL READING (SHEAR STRESS)
0
10
20
30
50
100
150
FANN RPM (SHEAR RATE)

FIGURES 2(a) - 2(d)

FIG. 2(a): THE LOCAL COORDINATE SYSTEM FOR A TYPICAL FINITE ELEMENT.

FIG. 2(b): THE LOCAL COORDINATES OF THE VERTICES OF A TYPICAL FINITE ELEMENT AND OF ONE OF ITS INTERIOR POINTS.

FIG. 2(c): A TYPICAL FINITE ELEMENT REFERRED TO A RECTILINEAR COORDINATE SYSTEM WITH EQUAL SCALES.

FIG. 2(d): A SIMPLE TWO ELEMENT MESH.

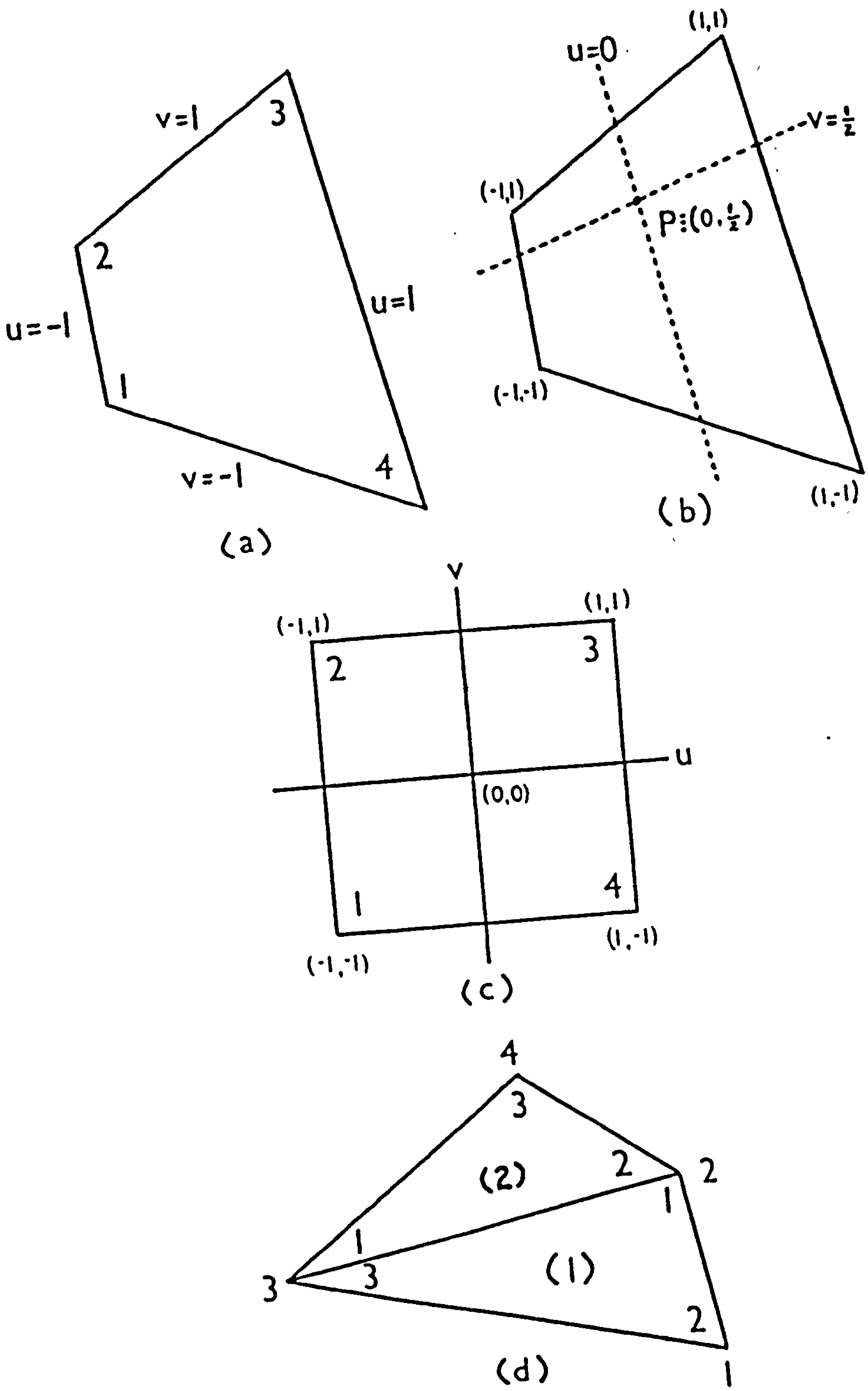
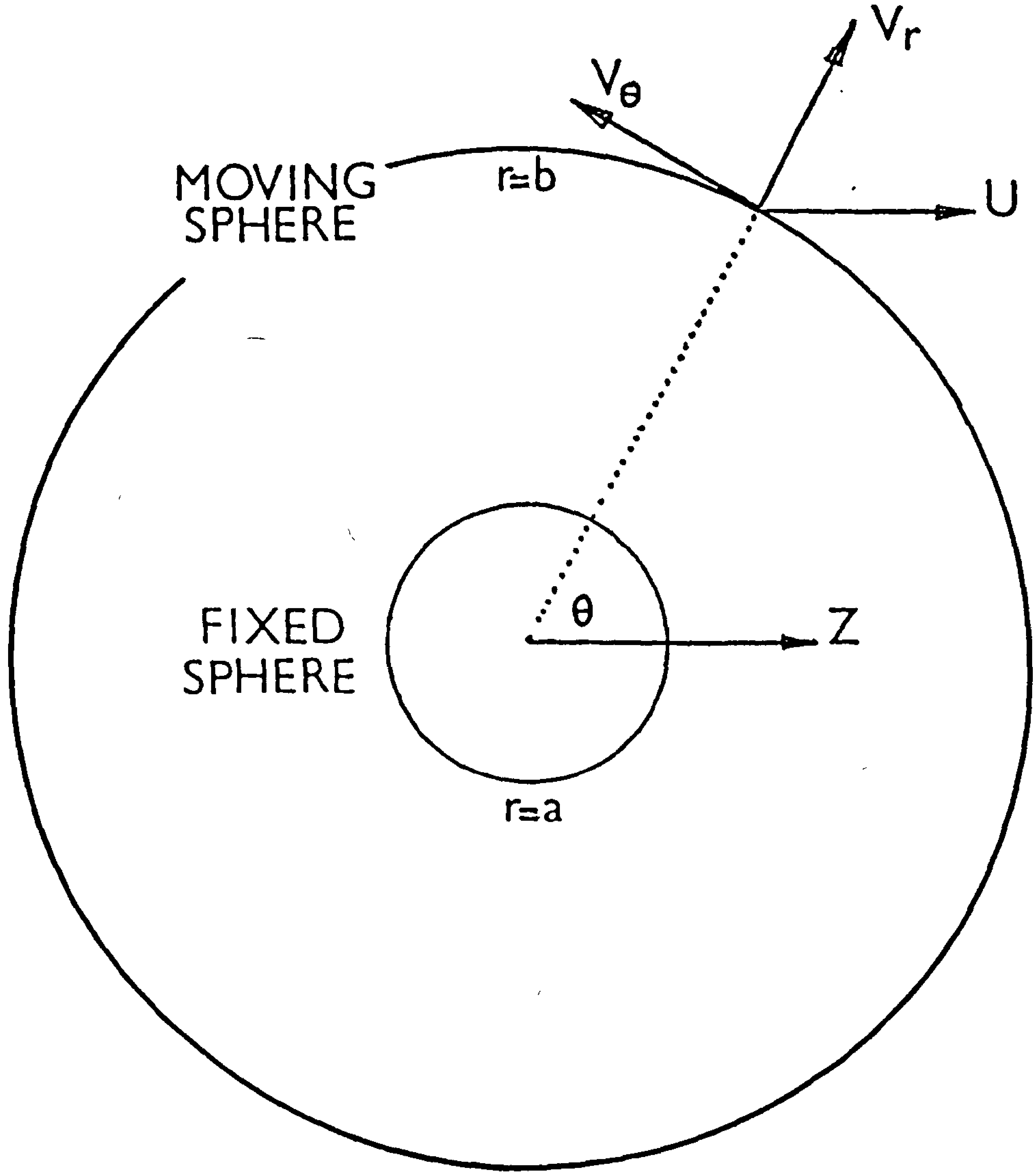

v=1
3
2
u=-1
u=1
1
v=-1
4
(a)
(1,1)
u=0
v=½
(-1,1)
P:(0,½)
(-1,-1)
(1,-1)
(b)
v
(-1,1)
(1,1)
2
3
u
(0,0)
1
4
(-1,-1)
(1,-1)
(c)
4
3
(2)
2
2
1
1
3
3
(1)
2
(d)
1

FIGURE 3

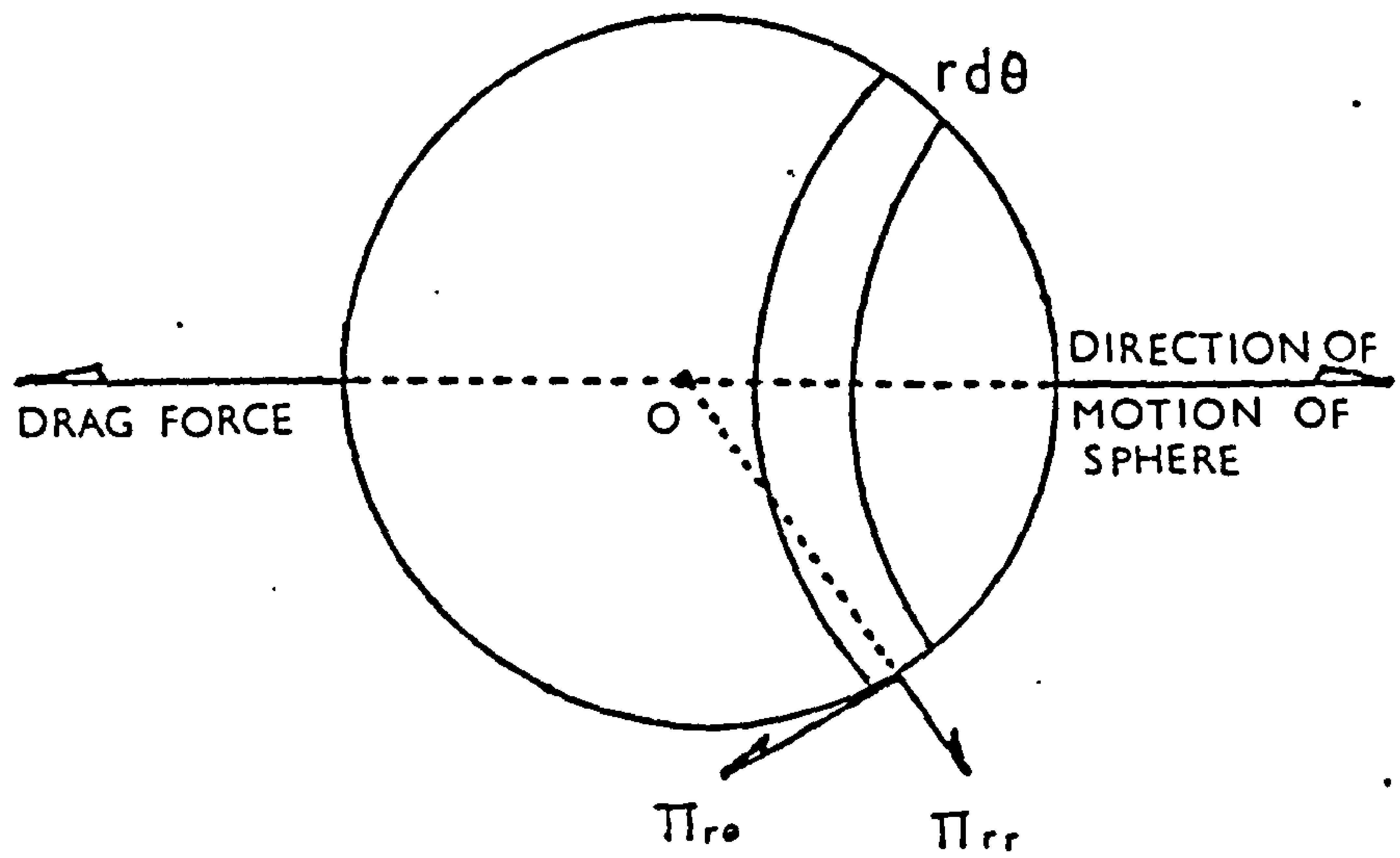
THIS FIGURE SHOWS A SCHEMATIC DIAGRAM FOR THE FLOW SITUATION WHEN THE OUTER SPHERE HAS A FINITE RADIUS.

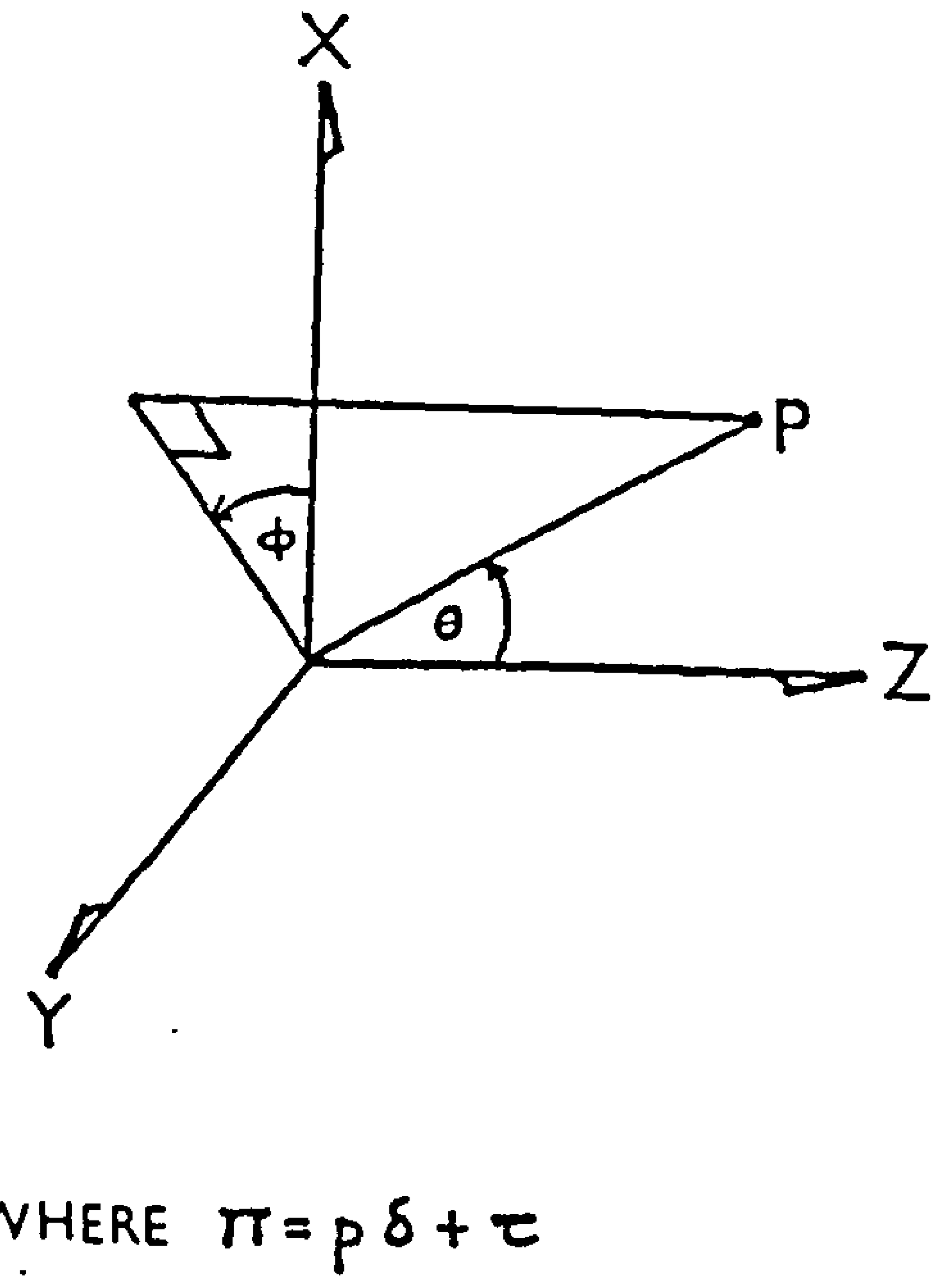


$V_r = U\cos\theta$ $\qquad$ $V_\theta = -U\sin\theta$

FIGURE 4

THIS FIGURE SHOWS THE ORIENTATION OF THE COMPONENTS OF THE TOTAL STRESS TENSOR AT THE SURFACE OF THE INNER SPHERE.

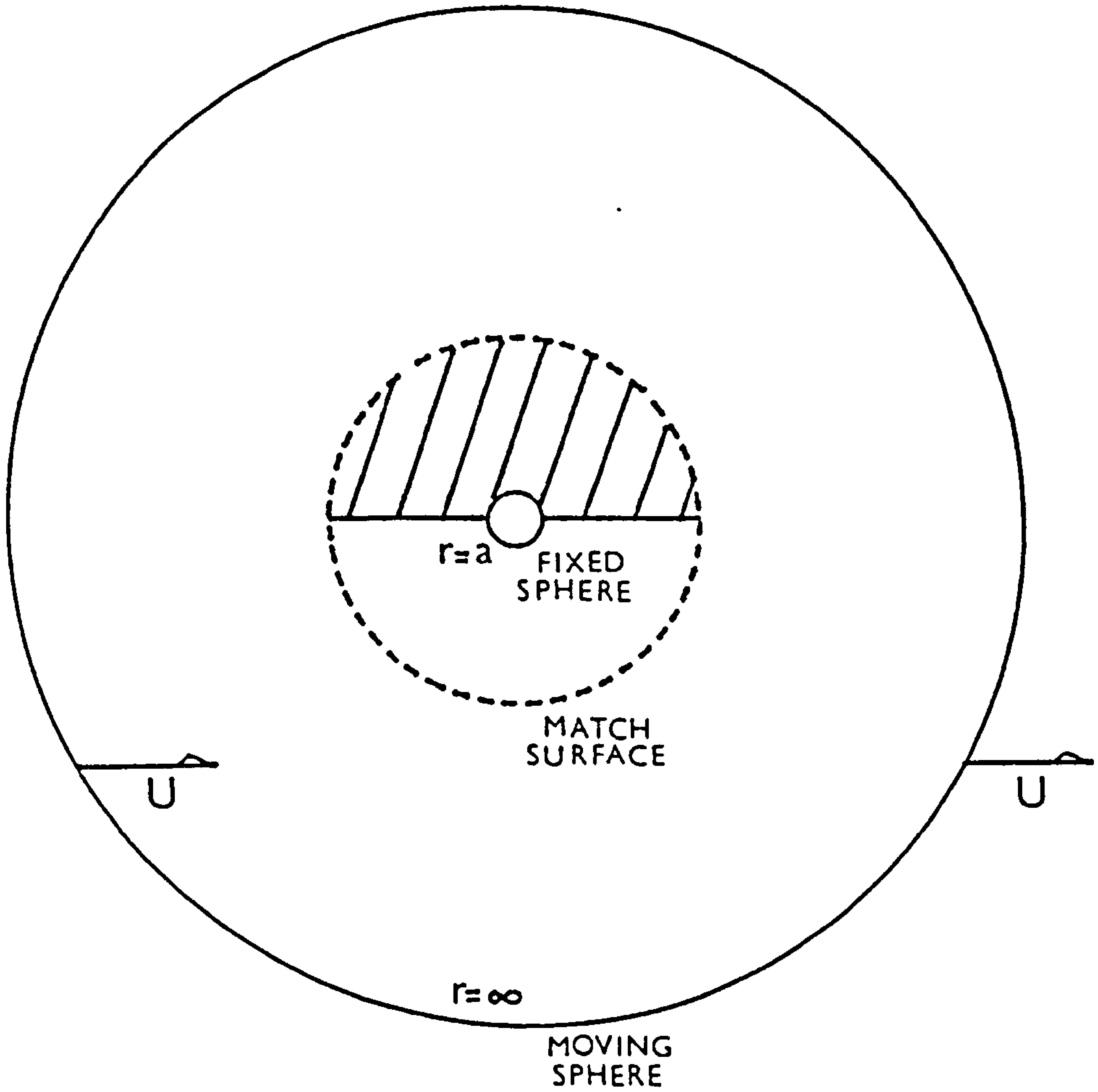


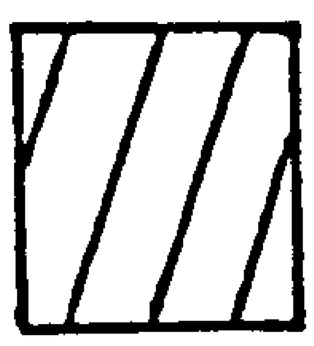


WHERE $\pi = p\delta + \tau$

FIGURE 5

THIS FIGURE SHOWS THE DOMAIN IN THE (r, θ) PLANE, OVER WHICH THE FINITE ELEMENT MESH IS CONSTRUCTED, WHEN THE OUTER SPHERE HAS AN 'INFINITE' RADIUS.




REGION OF FINITE ELEMENT MESH

THE FLUID FILLS THE VOLUME BETWEEN THE FIXED AND THE MOVING SPHERE

FIGURE 6

THIS FIGURE SHOWS A FINITE ELEMENT MESH IN THE (r, θ) DOMAIN WHEN THE OUTER SPHERE HAS AN 'INFINITE' RADIUS. THE BROKEN LINE SHOWS THE POSITION OF THE MATCHING SURFACE THE RADIUS OF WHICH IS 10 OR 20 TIMES THAT OF THE INNER SPHERE.

FINITE ELEMENT MESH

ANALYTICAL SOLUTION KNOWN FOR LARGE RADIUS

OMEGA AND PSI MATCHED AT SUFFICIENTLY LARGE RADIUS

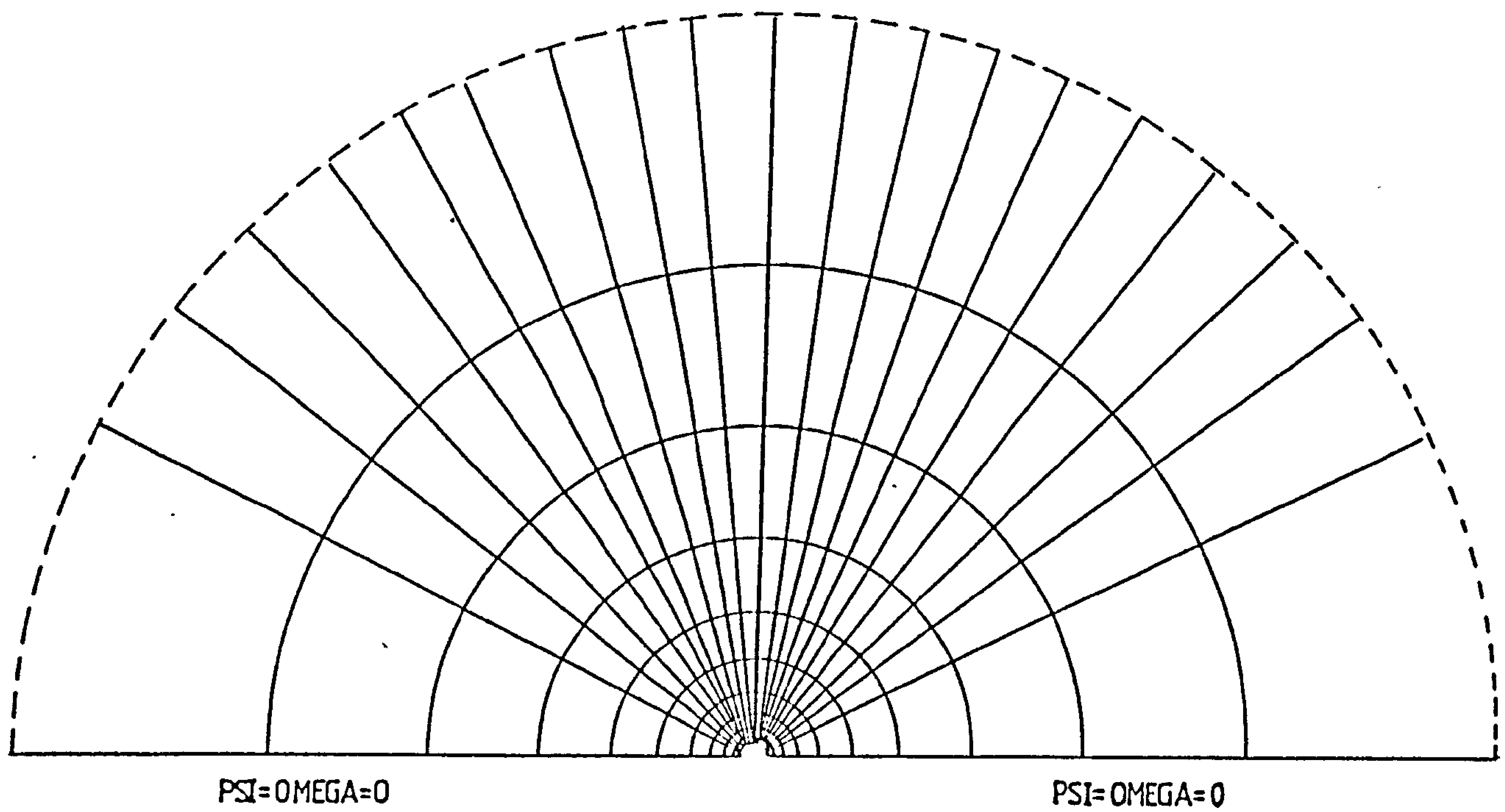


FIGURE 7

THIS FIGURE SHOWS THE ITERATIVE PROCEDURE USED TO CALCULATE AND PRINT F'(n) WHEN THE FLUID IS SLIGHTLY POWER LAW AND THE OUTER SPHERE HAS AN 'INFINITE' RADIUS.

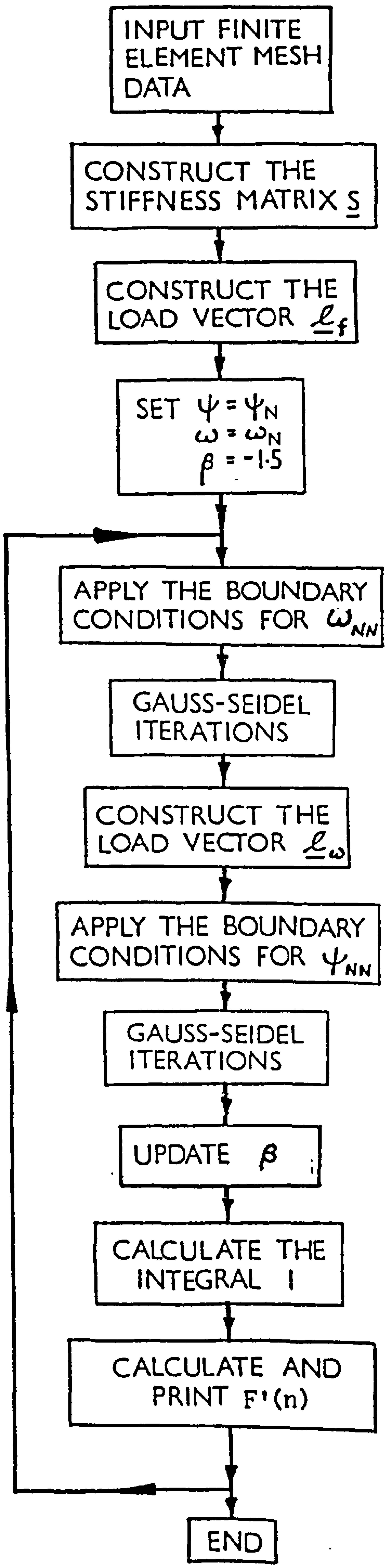

INPUT FINITE ELEMENT MESH DATA
CONSTRUCT THE STIFFNESS MATRIX $\underline{S}$
CONSTRUCT THE LOAD VECTOR $\underline{\ell}_f$
SET $\psi = \psi_N$ $\omega = \omega_N$ $\beta = -1{\cdot}5$
APPLY THE BOUNDARY CONDITIONS FOR ω_{NN}
GAUSS-SEIDEL ITERATIONS
CONSTRUCT THE LOAD VECTOR $\underline{\ell}_\omega$
APPLY THE BOUNDARY CONDITIONS FOR ψ_{NN}
GAUSS-SEIDEL ITERATIONS
UPDATE β
CALCULATE THE INTEGRAL I
CALCULATE AND PRINT $F'(n)$
END

FIGURE 8

THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN F(n,d) AND n FOR A SLIGHTLY POWER LAW FLUID WHEN d TAKES THE VALUES 2, 10, 20, 40 AND 100.

TWO SETS OF RESULTS ARE PRESENTED:

(i) THE FINITE ELEMENT RESULTS.

(ii) THE RESULTS OF LOCKYER, DAVIES AND JONES OBTAINED USING FINITE DIFFERENCE METHODS.

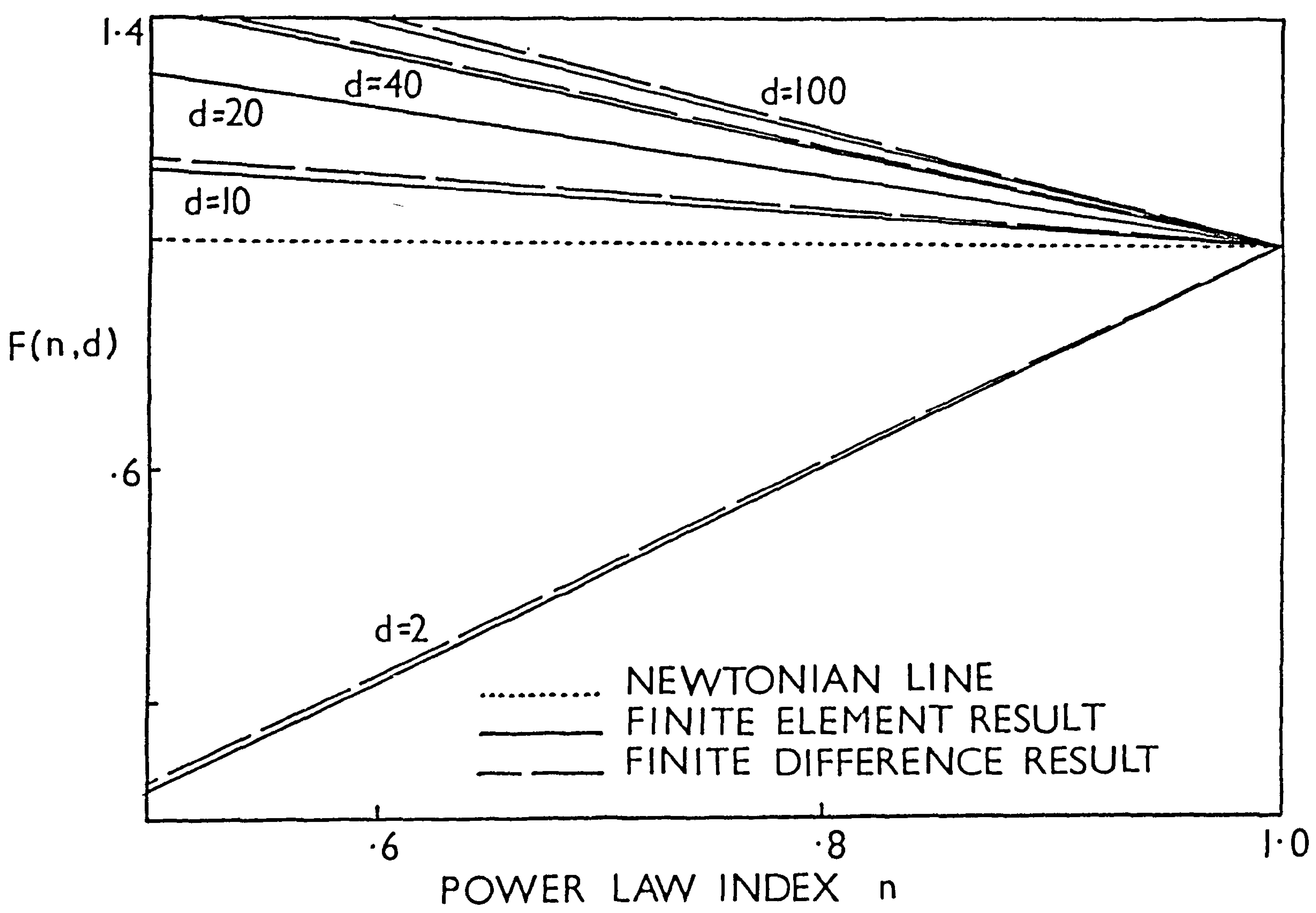
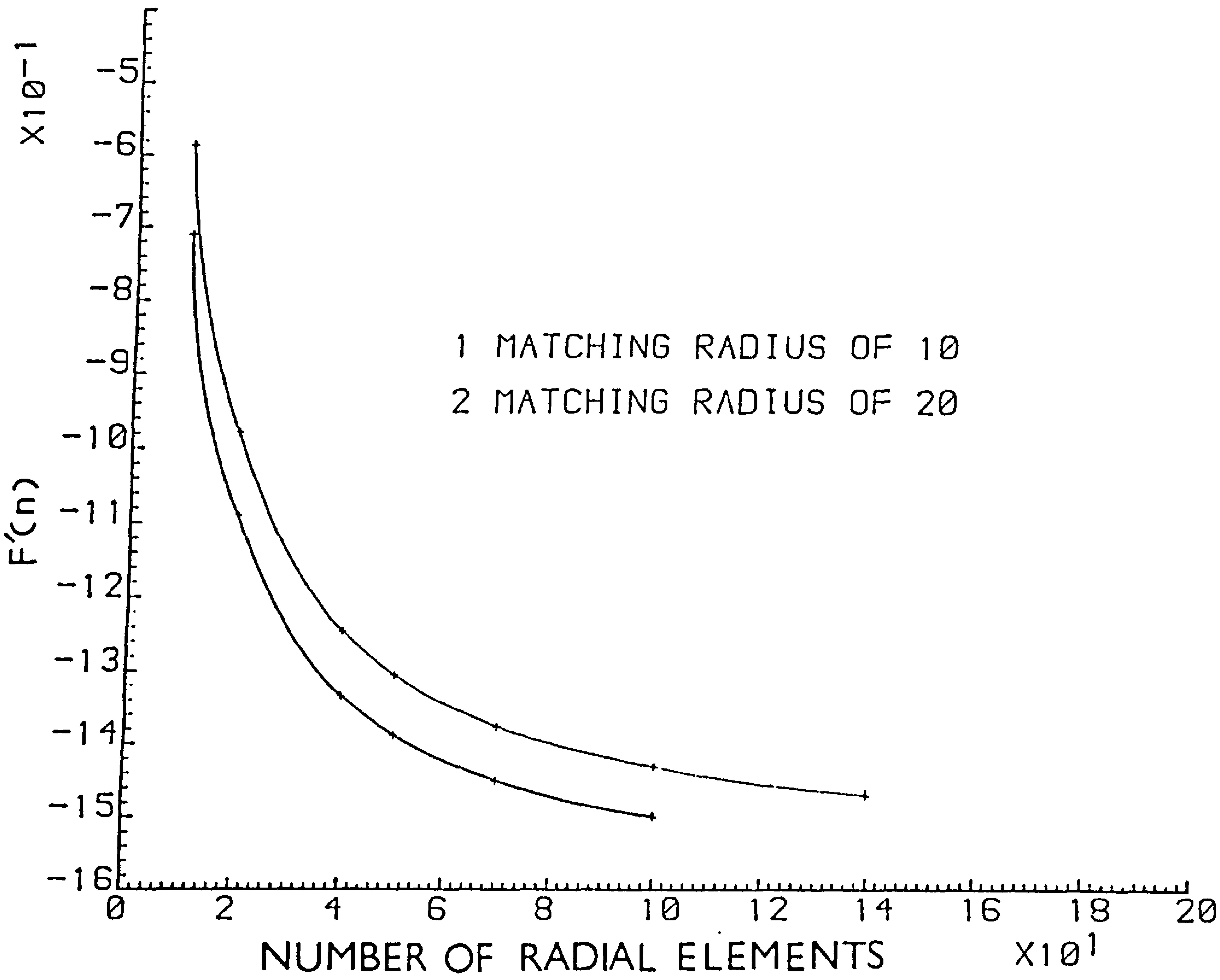

1·4
·6
F(n,d)
d=100
d=40
d=20
d=10
d=2
NEWTONIAN LINE
FINITE ELEMENT RESULT
FINITE DIFFERENCE RESULT
·6
·8
1·0
POWER LAW INDEX n

FIGURE 9

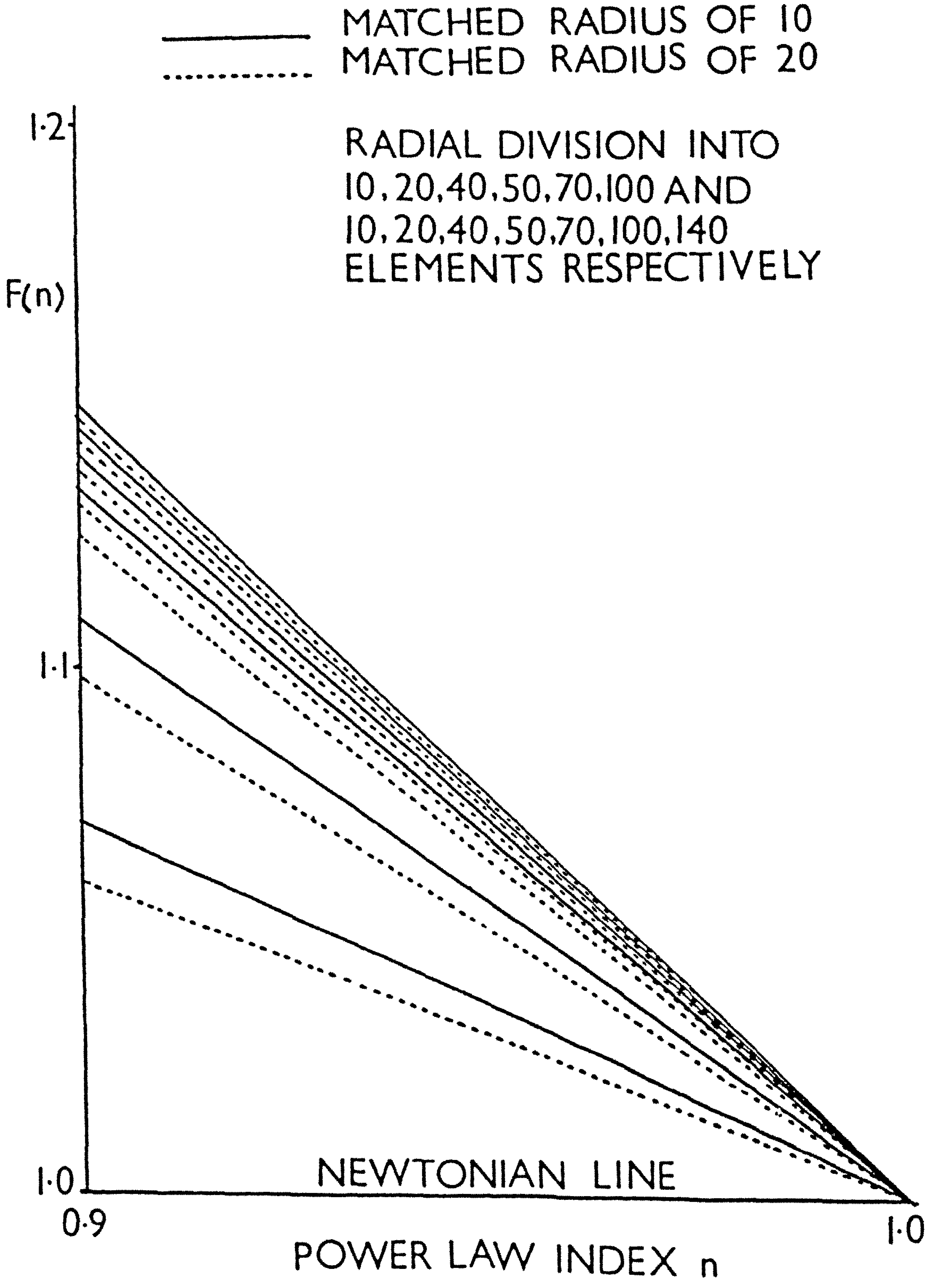
THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN F'(n) AND THE NUMBER OF RADIAL ELEMENTS USED IN THE FINITE ELEMENT MESH. THE MATCHING SURFACE USED HAD A RADIUS OF EITHER 10 OR 20 TIMES THAT OF THE INNER SPHERE.



X10⁻¹
F′(n)
-5
-6
-7
-8
-9
-10
-11
-12
-13
-14
-15
-16
0
2
4
6
8
10
12
14
16
18
20
NUMBER OF RADIAL ELEMENTS
X10¹
1 MATCHING RADIUS OF 10
2 MATCHING RADIUS OF 20

FIGURE 10

THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN $F(n)$ AND n FOR VARIOUS NUMBERS OF RADIAL ELEMENTS USED IN THE FINITE ELEMENT MESH. THE MATCHING SURFACE USED HAD A RADIUS OF EITHER 10 OR 20 TIMES THAT OF THE INNER SPHERE.



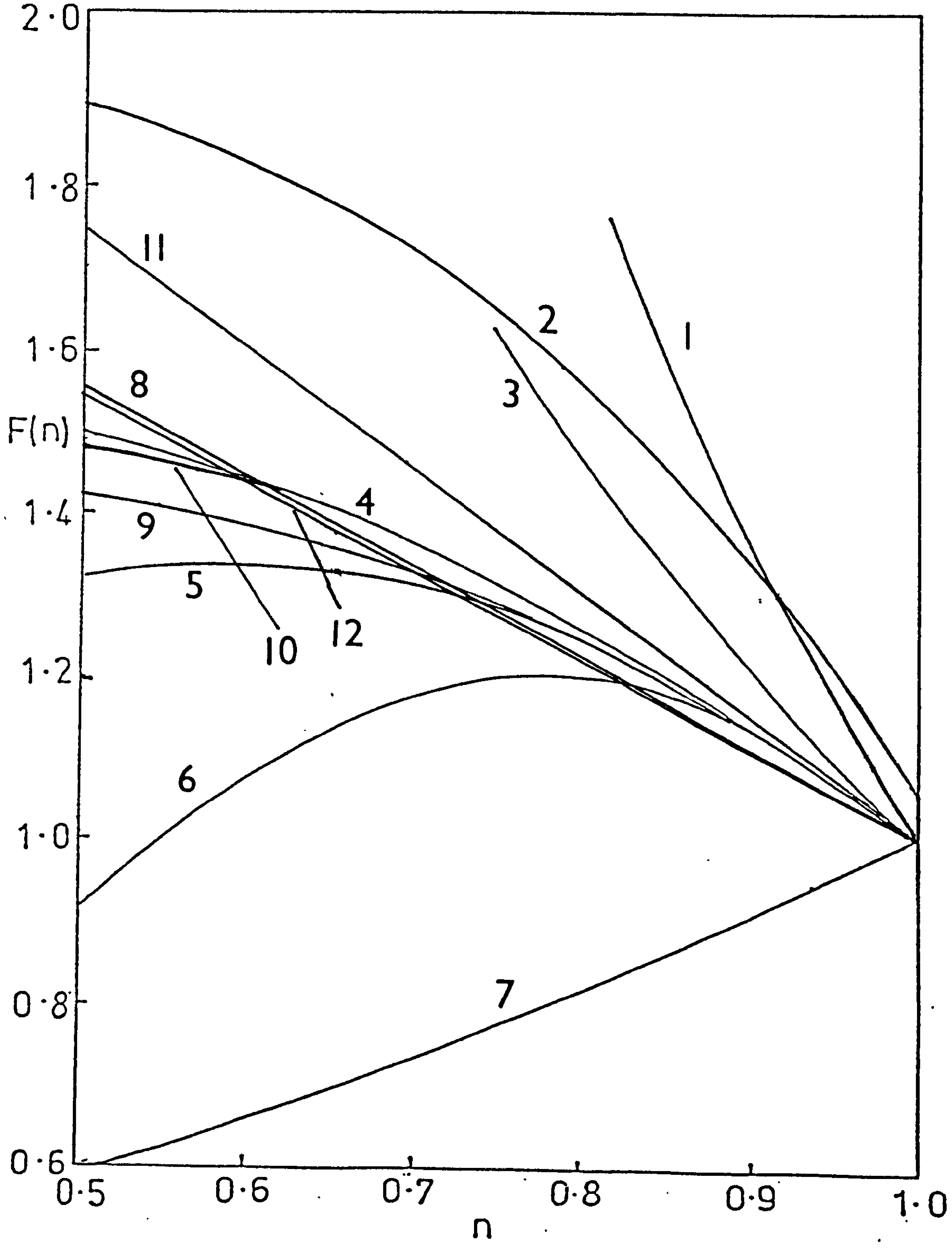
MATCHED RADIUS OF 10
MATCHED RADIUS OF 20
RADIAL DIVISION INTO 10,20,40,50,70,100 AND 10,20,40,50,70,100,140 ELEMENTS RESPECTIVELY
F(n)
1·2
1·1
1·0
NEWTONIAN LINE
0·9
1·0
POWER LAW INDEX n

FIGURE 11

THIS FIGURE SHOWS A COMPARISON OF THE RESULTS OBTAINED FOR THE DRAG CORRECTION FACTOR, F(n), WITH THE RESULTS OF OTHER WORKERS.

CURVE

1,3	J.C. SLATTERY[48]
2	Y. TOMITA[49]
4,6	M.L. WASSERMAN, J.C. SLATTERY[25]
4,5	Y.I. CHO AND J.P. HARTNETT[28]
7	A. FARAROURI, R.C. KINTNER[50]
8	M.A. LOCKYER, J.M. DAVIES AND T.E.R. JONES[37]
9	G.U. DAZHI AND R.J. TANNER[31]
10	M.J. CROCHET, A.R. DAVIES AND K. WALTERS[30]
11	FINITE ELEMENT RESULT (OUTER SPHERE 'INFINITE')
12	FINITE ELEMENT RESULT (d = 100)

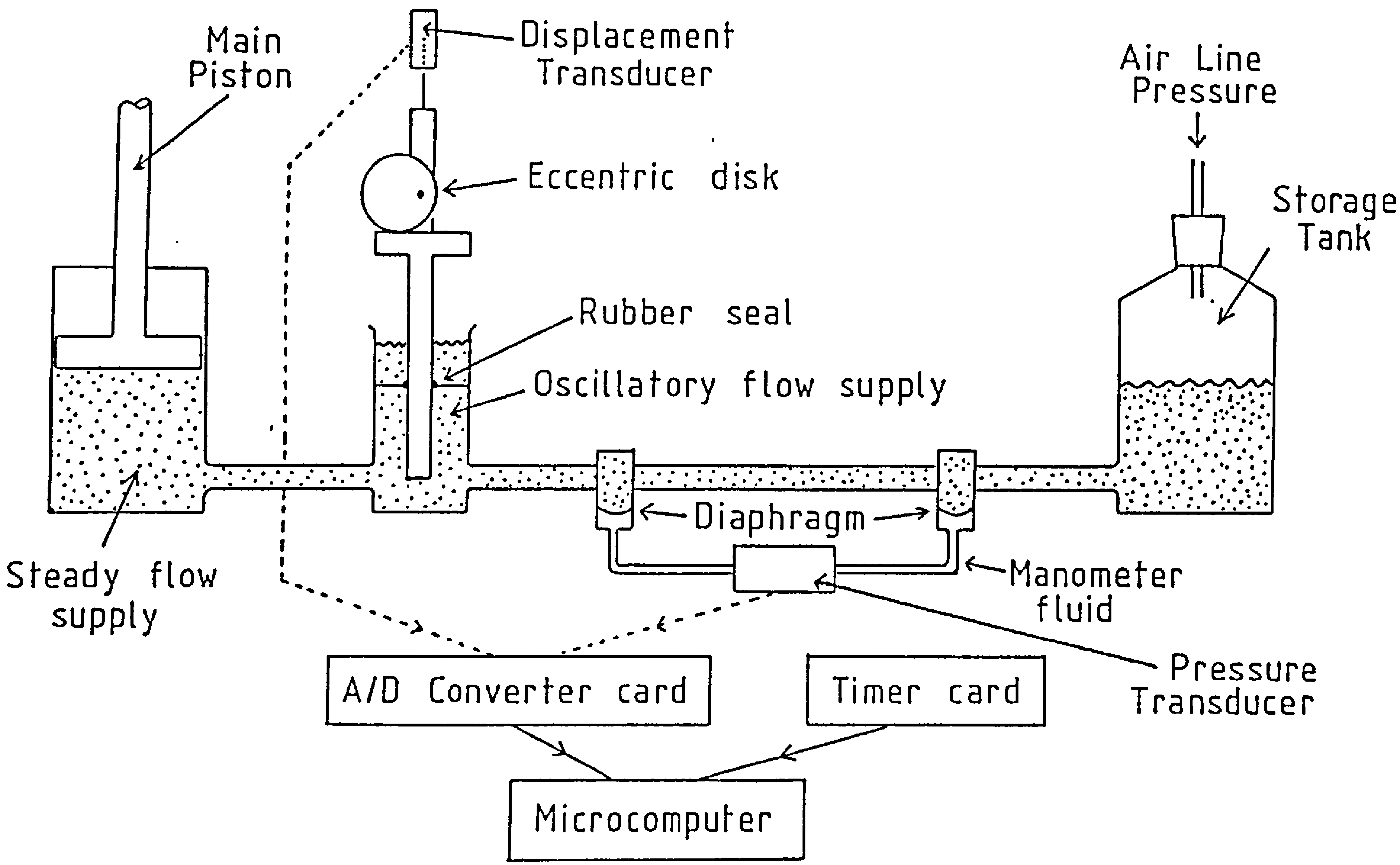


2·0
1·8
1·6
F(n)
1·4
1·2
1·0
0·8
0·6
0·5
0·6
0·7
0·8
0·9
1·0
n
1
2
3
4
5
6
7
8
9
10
11
12

FIGURE 12

THIS FIGURE SHOWS A SCHEMATIC DIAGRAM OF THE APPARATUS OF CHAKRABATI AND DAVIES[45] IN WHICH THE MEAN FLOW RATE AND THE PULSATILE FLOW RATE ARE CONTROLLED AND THE PRESSURE GRADIENT IS MEASURED.

CONTROLLED PULSATILE FLOW APPARATUS



FIGURES 13 - 16

THESE FIGURES SHOW THE VARIATION OF THE NORMALISED MEAN PRESSURE GRADIENT WITH F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. THE VALUE OF EPSILON IS 0.5, 1, 5 OR 10. THE CONTINUOUS LINES REFER TO NUMERICAL DATA, THE BROKEN LINES TO RESULTS OBTAINED FROM PERTUBATION THEORY AND THE DOTTED LINES TO THE RESULTS WHEN FLUID INERTIA IS ZERO.

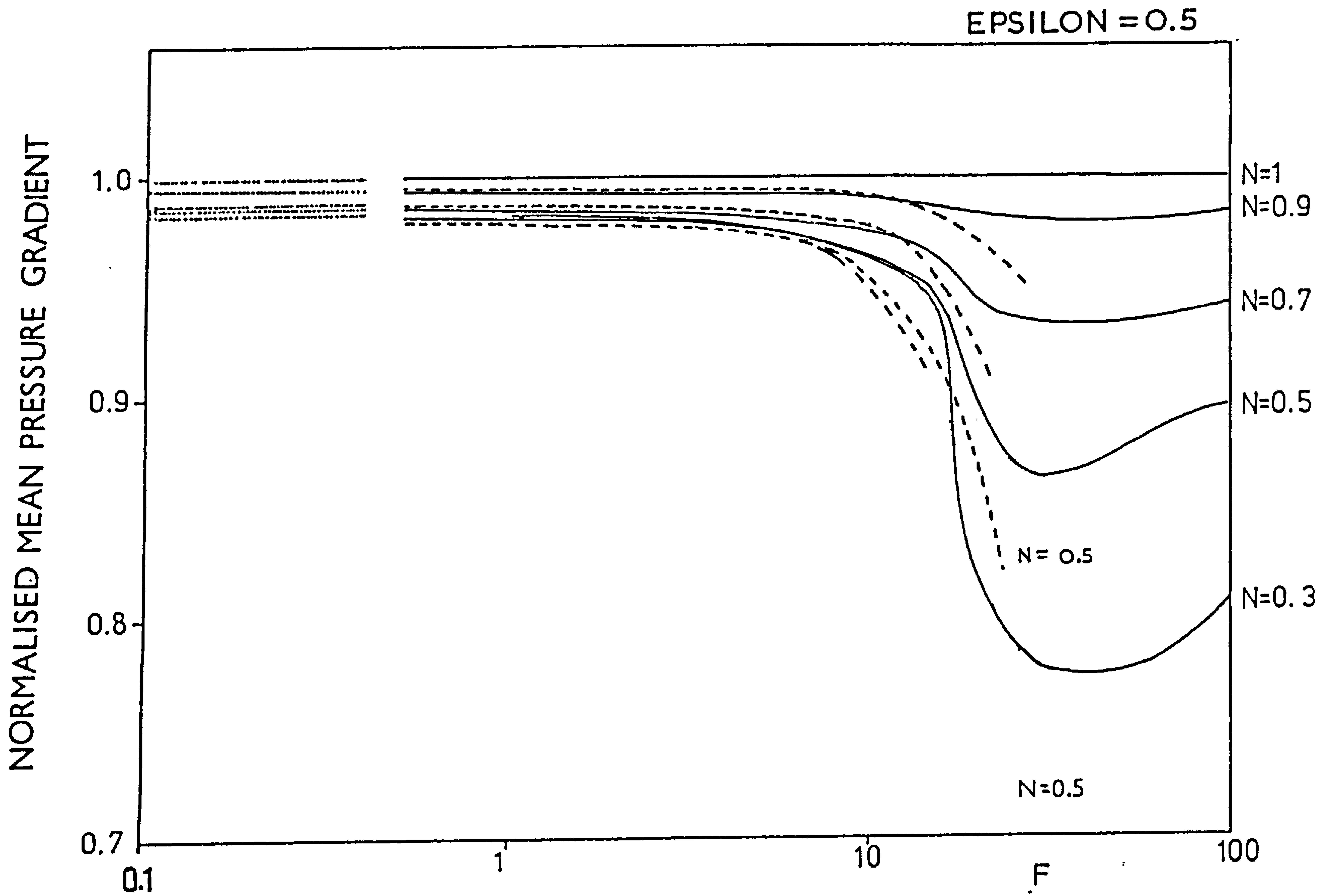

EPSILON = 0.5
NORMALISED MEAN PRESSURE GRADIENT
1.0
0.9
0.8
0.7
0.1
1
10
100
F
N=1
N=0.9
N=0.7
N=0.5
N=0.3
N= 0.5
N=0.5

FIGURE 13

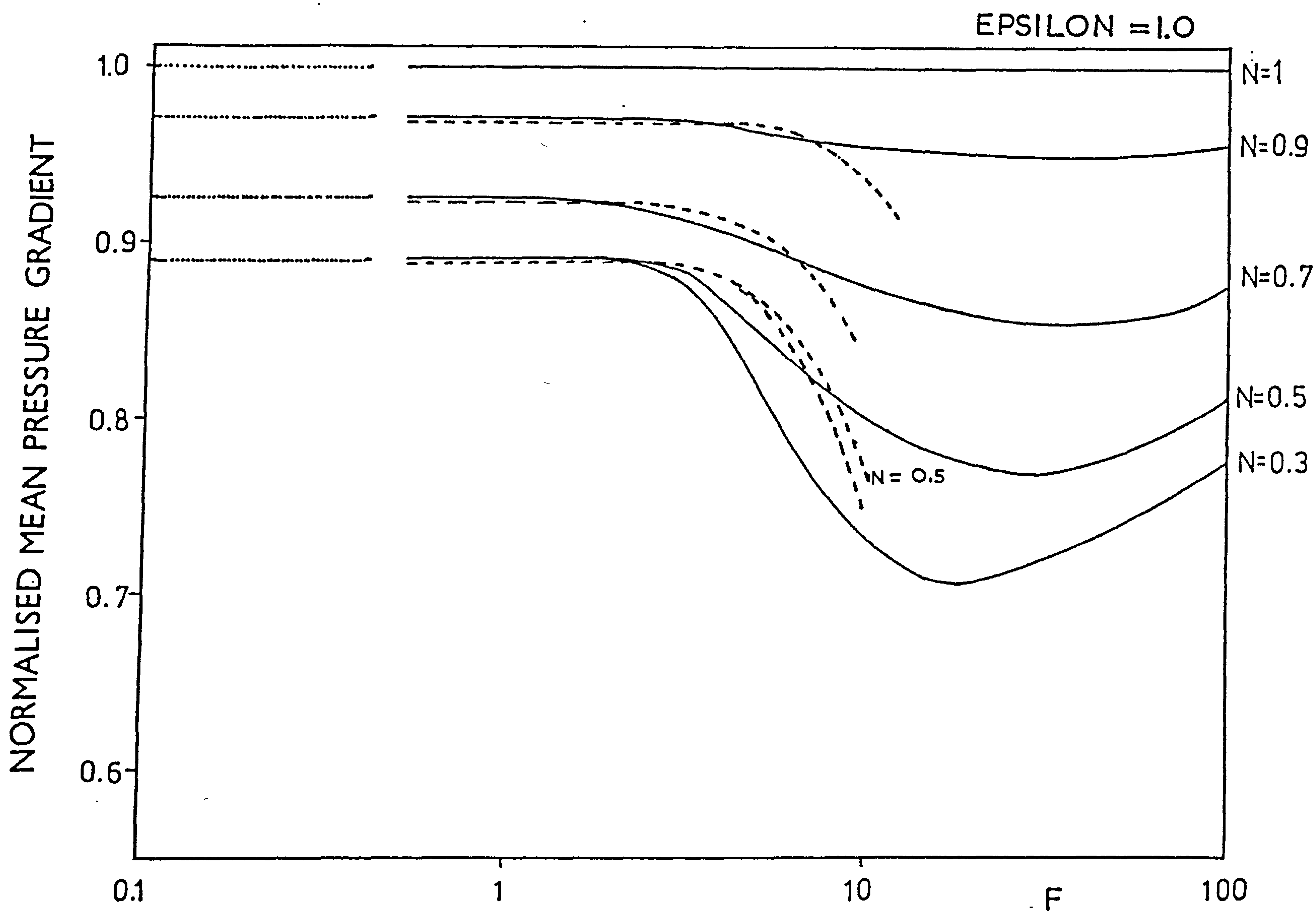
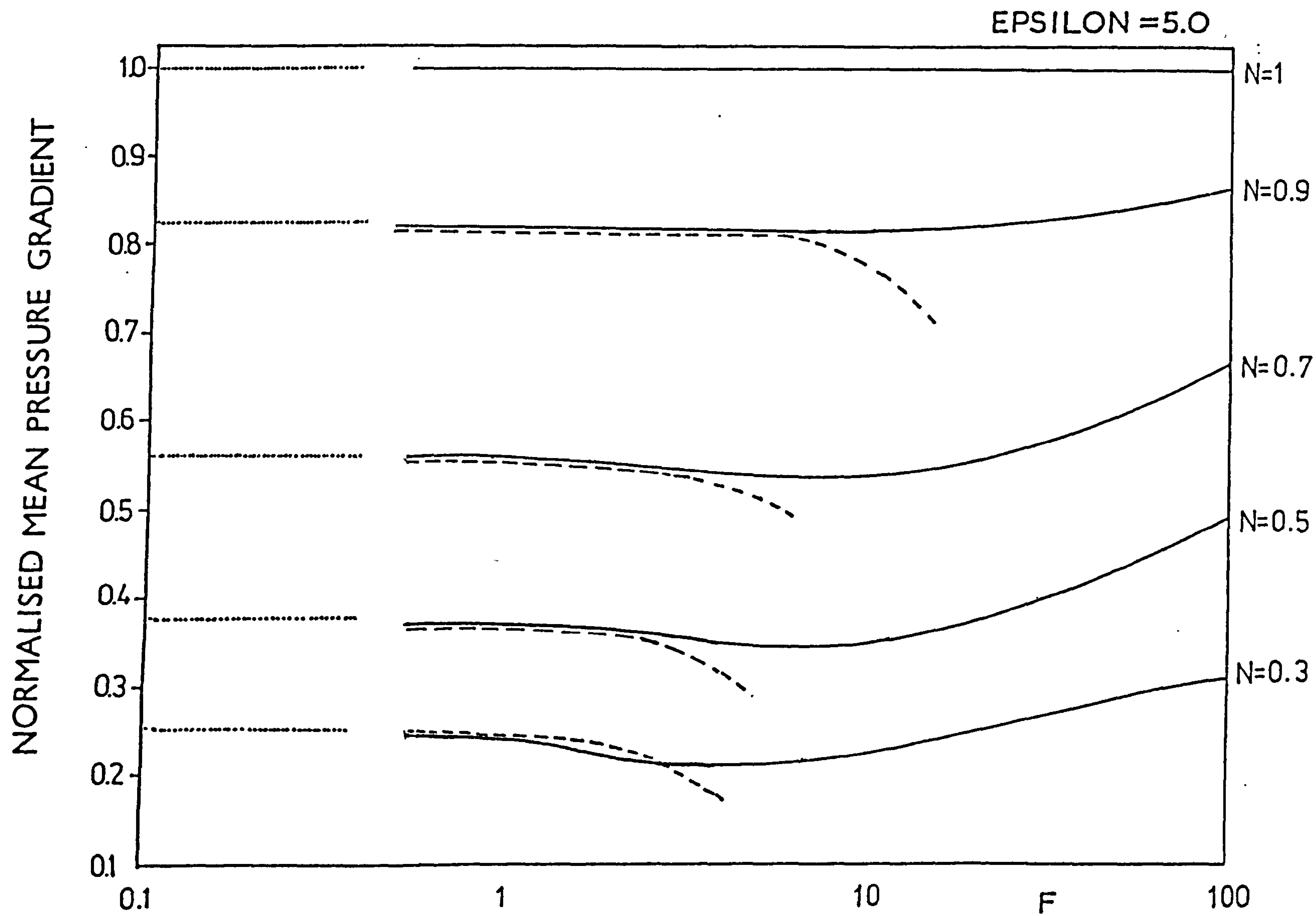

EPSILON =1.0
NORMALISED MEAN PRESSURE GRADIENT
1.0
0.9
0.8
0.7
0.6
0.1
1
10
100
F
N=1
N=0.9
N=0.7
N=0.5
N=0.3
N = 0.5

FIGURE 14



EPSILON =5.0
NORMALISED MEAN PRESSURE GRADIENT
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1
1
10
F
100
N=1
N=0.9
N=0.7
N=0.5
N=0.3

FIGURE 15

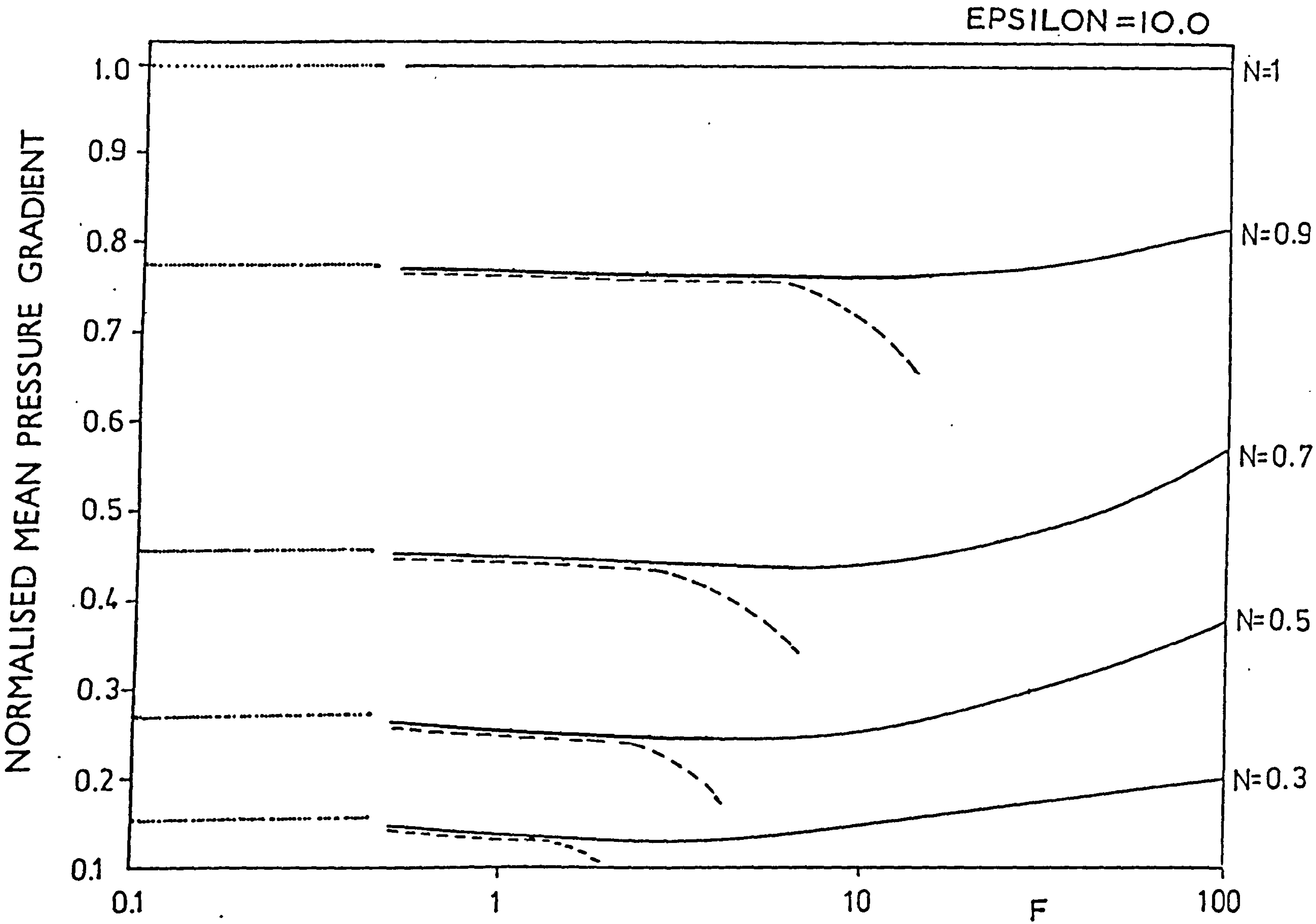
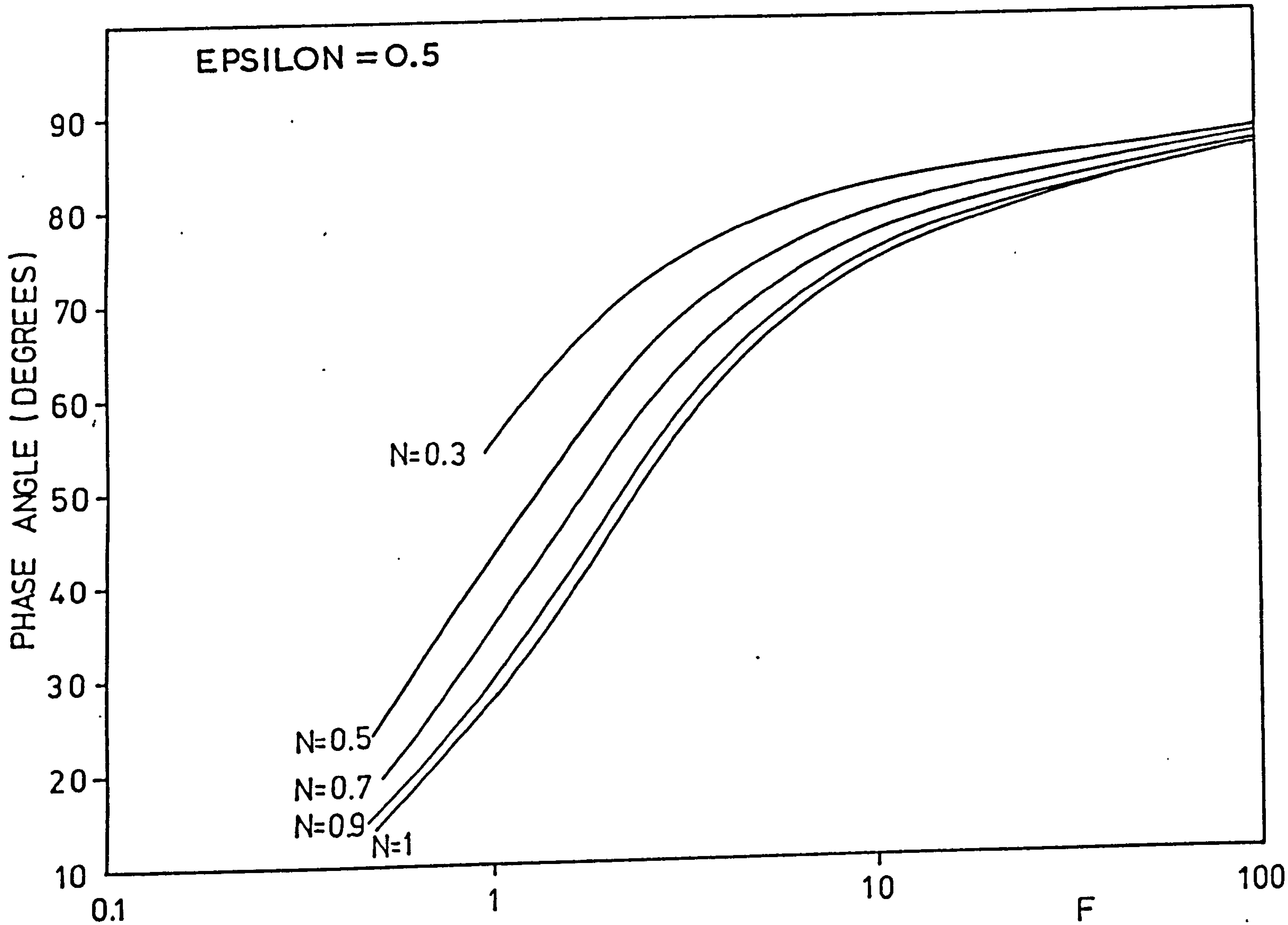

EPSILON=10.0
NORMALISED MEAN PRESSURE GRADIENT
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1
1
10
100
F
N=1
N=0.9
N=0.7
N=0.5
N=0.3

FIGURE 16

FIGURES 17 - 20

THESE FIGURES SHOW THE VARIATION OF THE PHASE ANGLE WITH F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. THE VALUE OF EPSILON IS 0.5, 1, 5 OR 10.



EPSILON = 0.5
PHASE ANGLE (DEGREES)
90
80
70
60
50
40
30
20
10
N=0.3
N=0.5
N=0.7
N=0.9
N=1
0.1
1
10
100
F

FIGURE 17

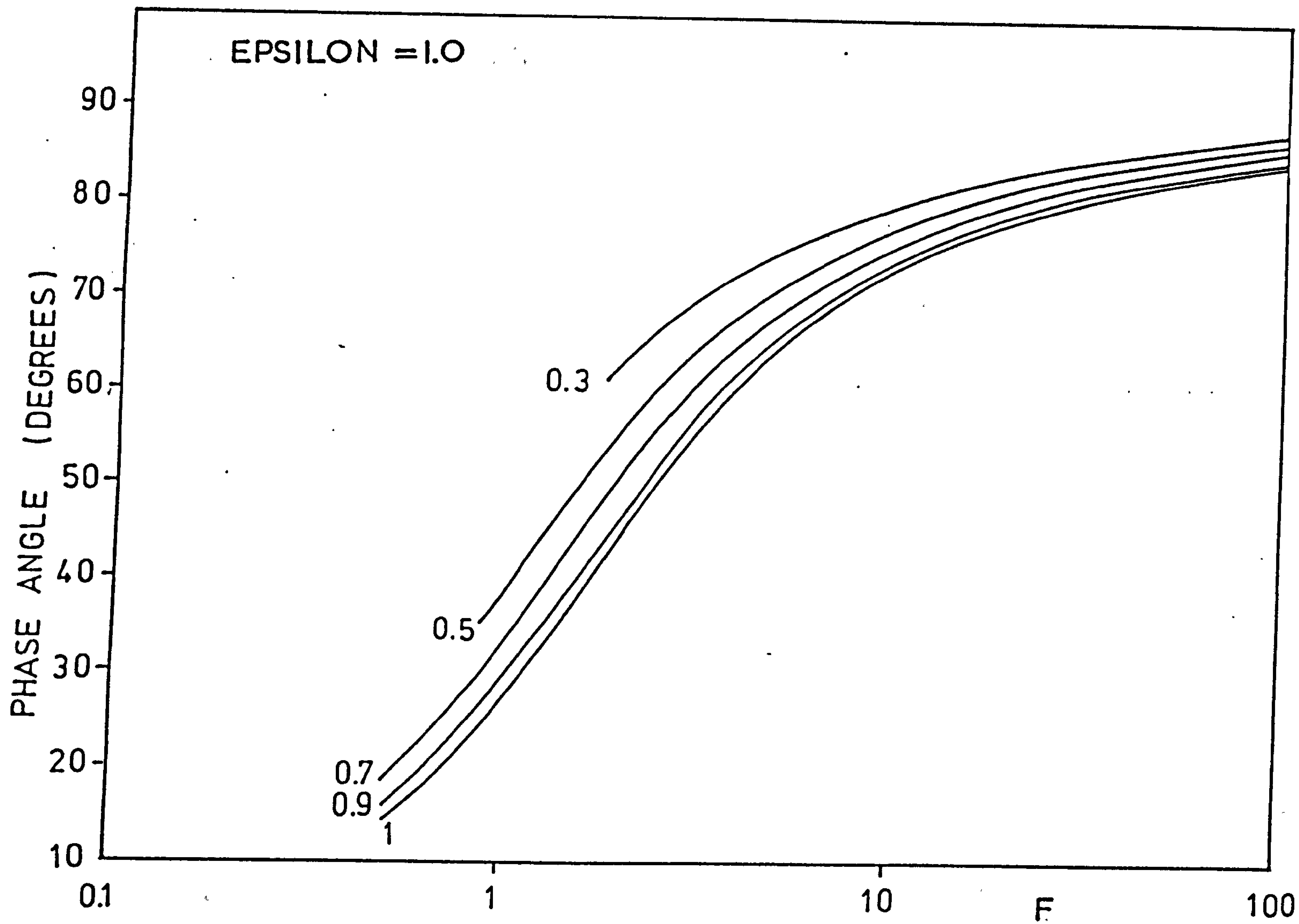

EPSILON =1.0
PHASE ANGLE (DEGREES)
90
80
70
60
50
40
30
20
10
0.1
1
10
100
F
0.3
0.5
0.7
0.9
1

FIGURE 18

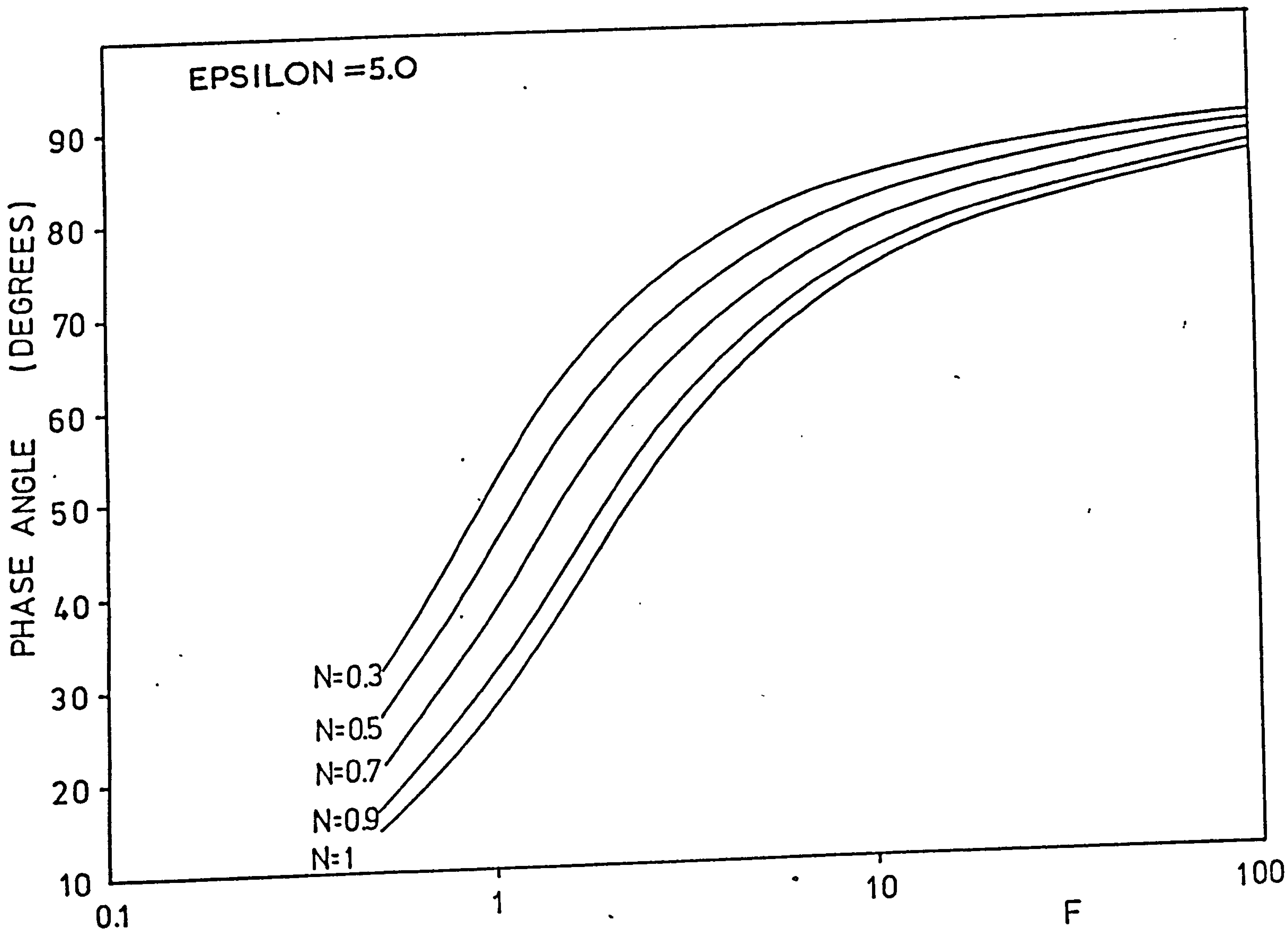

EPSILON =5.0
PHASE ANGLE (DEGREES)
90
80
70
60
50
40
30
20
10
N=0.3
N=0.5
N=0.7
N=0.9
N=1
0.1
1
10
100
F

FIGURE 19

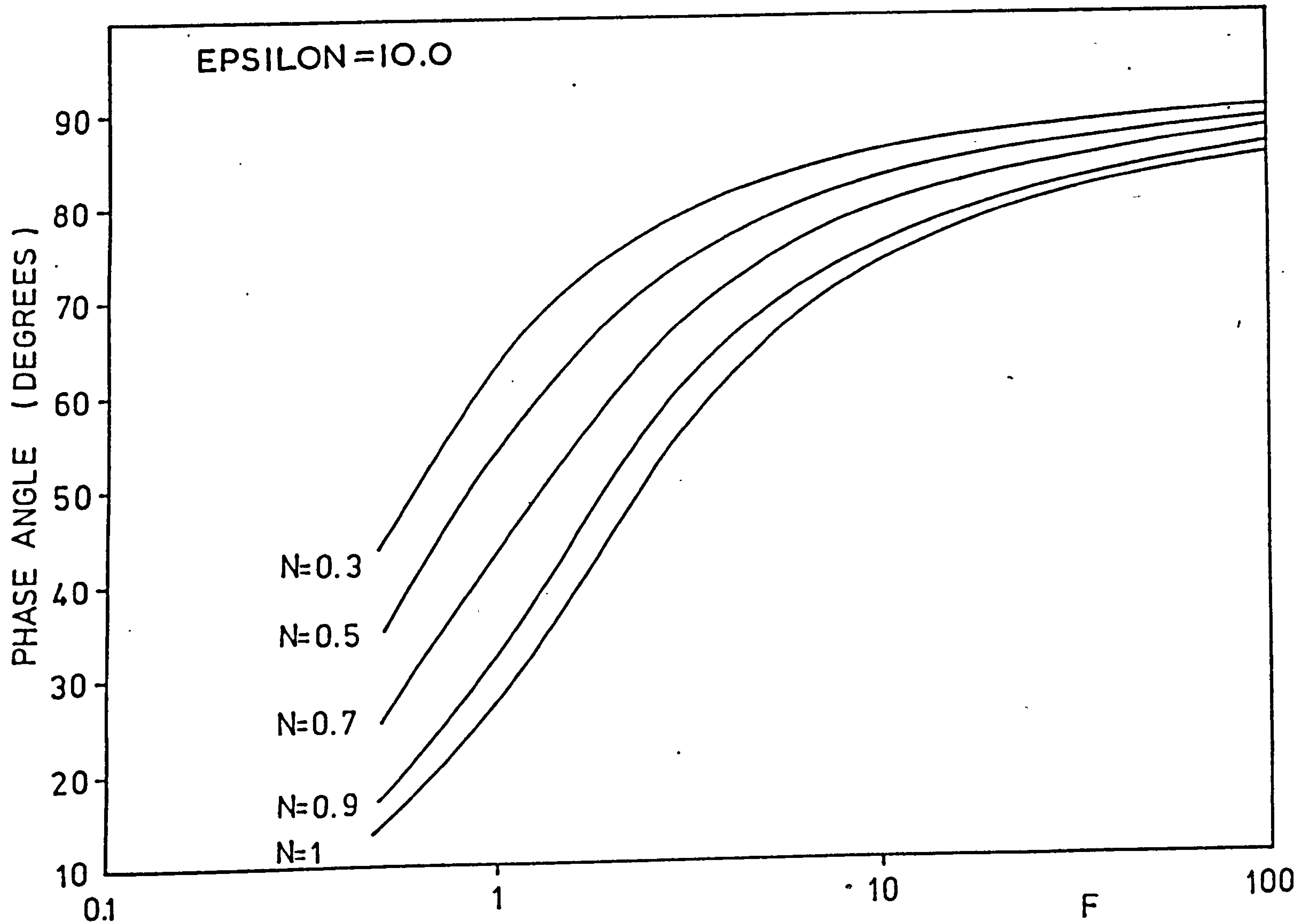

EPSILON=10.0
PHASE ANGLE (DEGREES)
90
80
70
60
50
40
30
20
10
N=0.3
N=0.5
N=0.7
N=0.9
N=1
0.1
1
10
100
F

FIGURE 20

FIGURES 21-24

THESE FIGURES SHOW THE VARIATION OF THE FUNDAMENTAL PRESSURE AMPLITUDE WITH F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. THE VALUE FOR EPSILON IS 0.5, 1, 5 OR 10. THE SLOPE OF THE GRAPH TENDS TO 1 FOR HIGH VALUES OF F FOR ALL EPSILON.

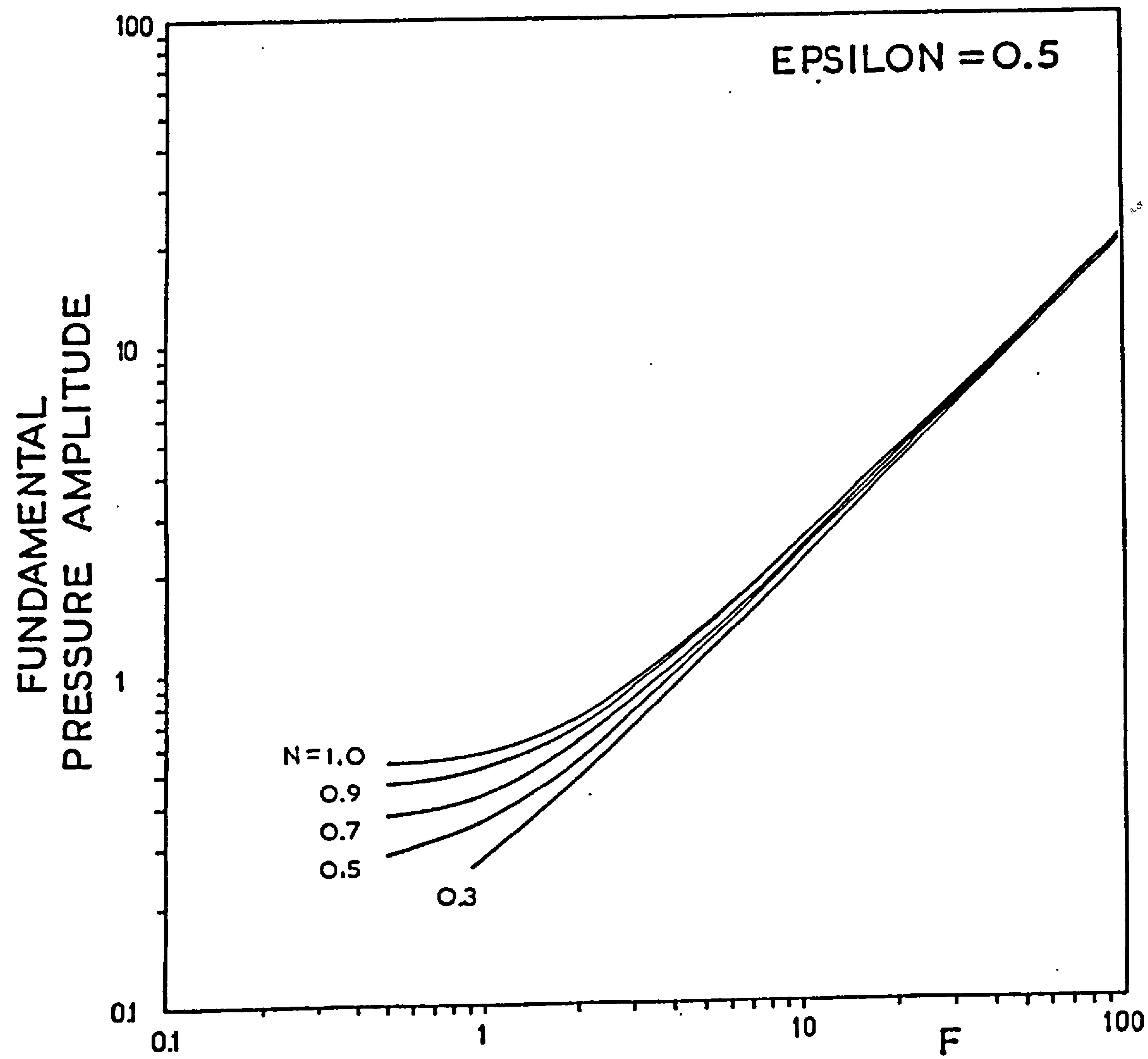

EPSILON = 0.5
FUNDAMENTAL
PRESSURE AMPLITUDE
100
10
1
0.1
0.1
1
10
100
F
N=1.0
0.9
0.7
0.5
0.3

FIGURE 21

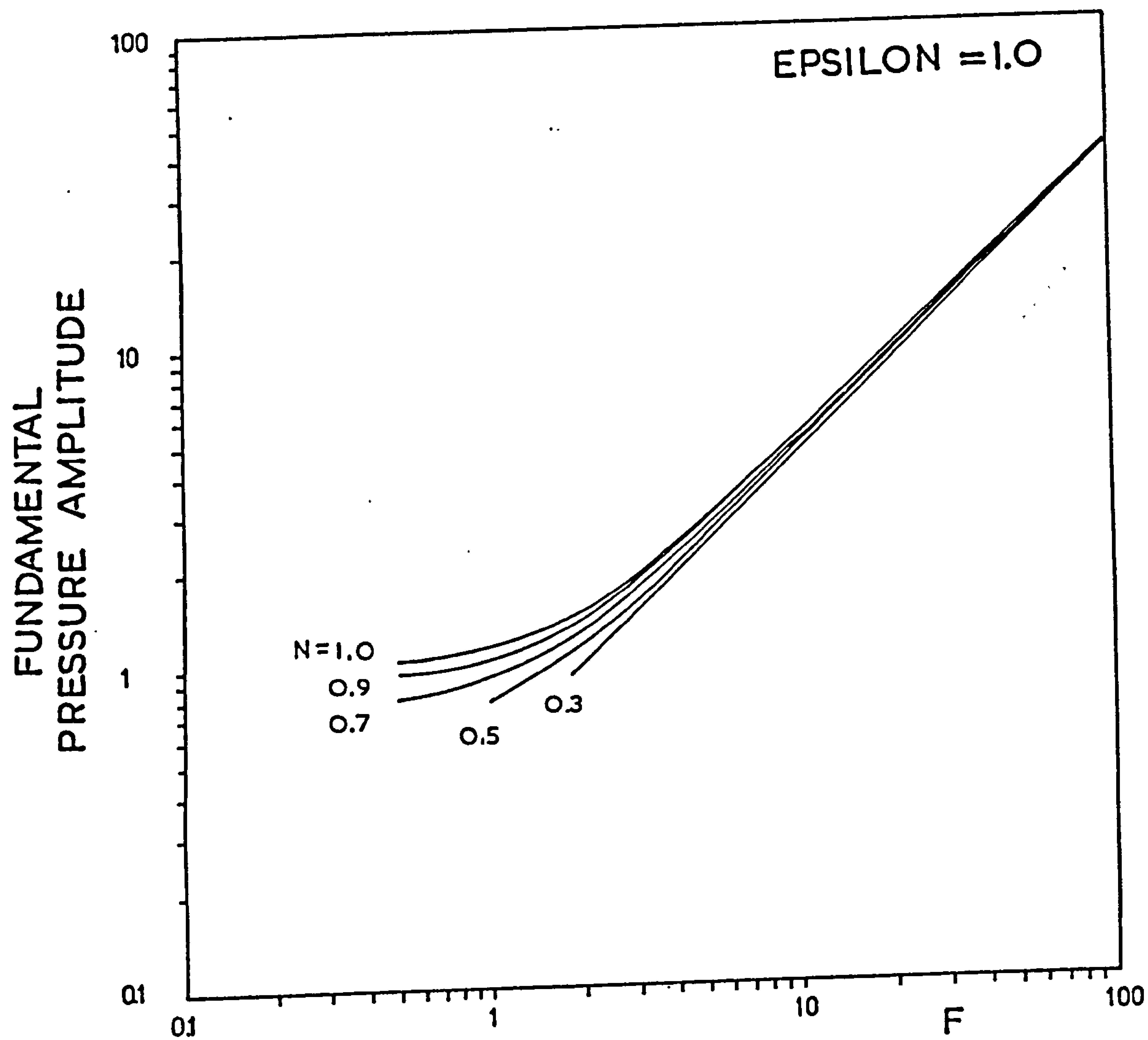

EPSILON =1.0
FUNDAMENTAL PRESSURE AMPLITUDE
100
10
1
0.1
0.1
1
10
100
F
N=1.0
0.9
0.7
0.5
0.3

FIGURE 22

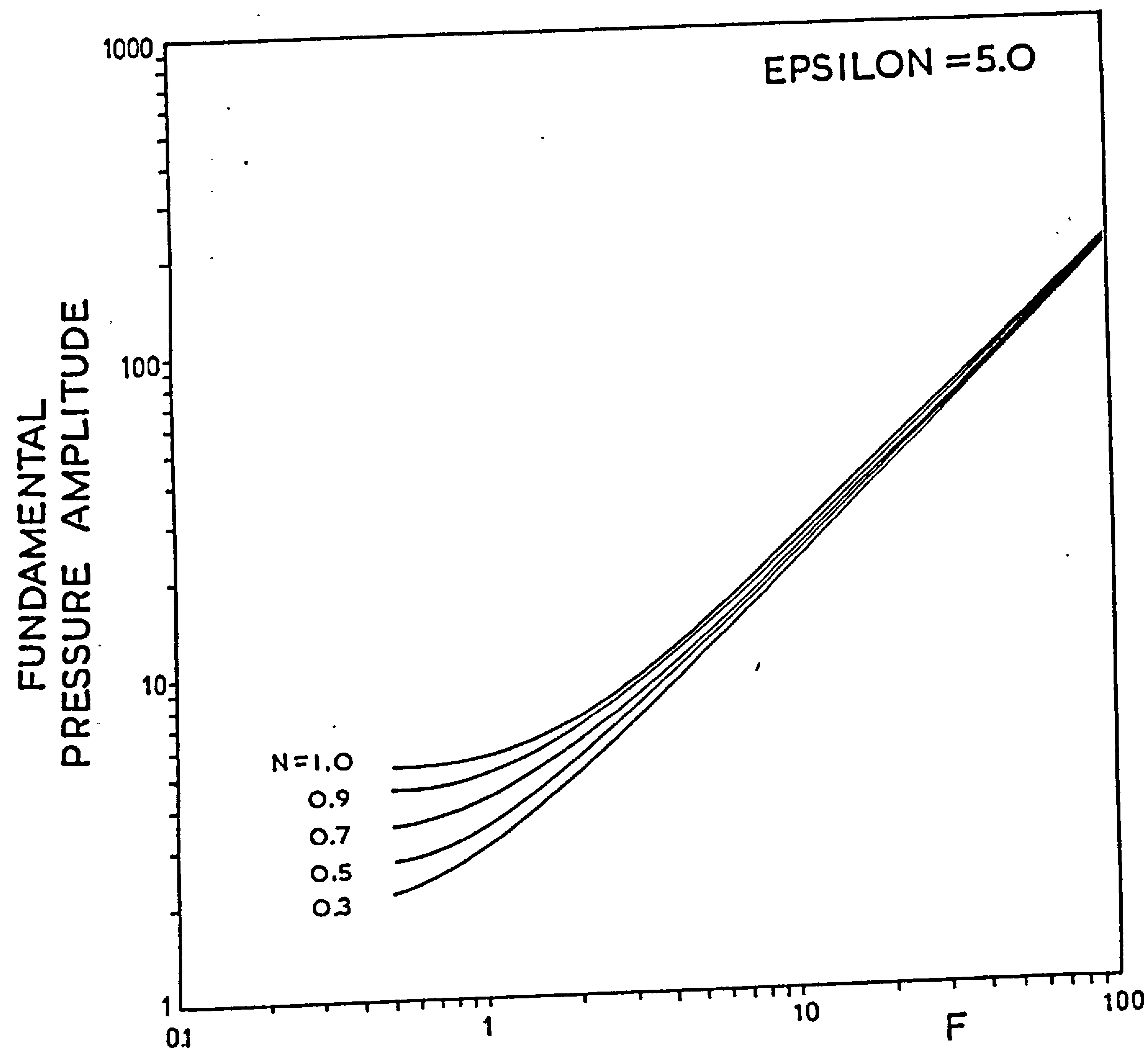

EPSILON =5.0
FUNDAMENTAL
PRESSURE AMPLITUDE
1000
100
10
1
N=1.0
0.9
0.7
0.5
0.3
0.1
1
10
100
F

FIGURE 23

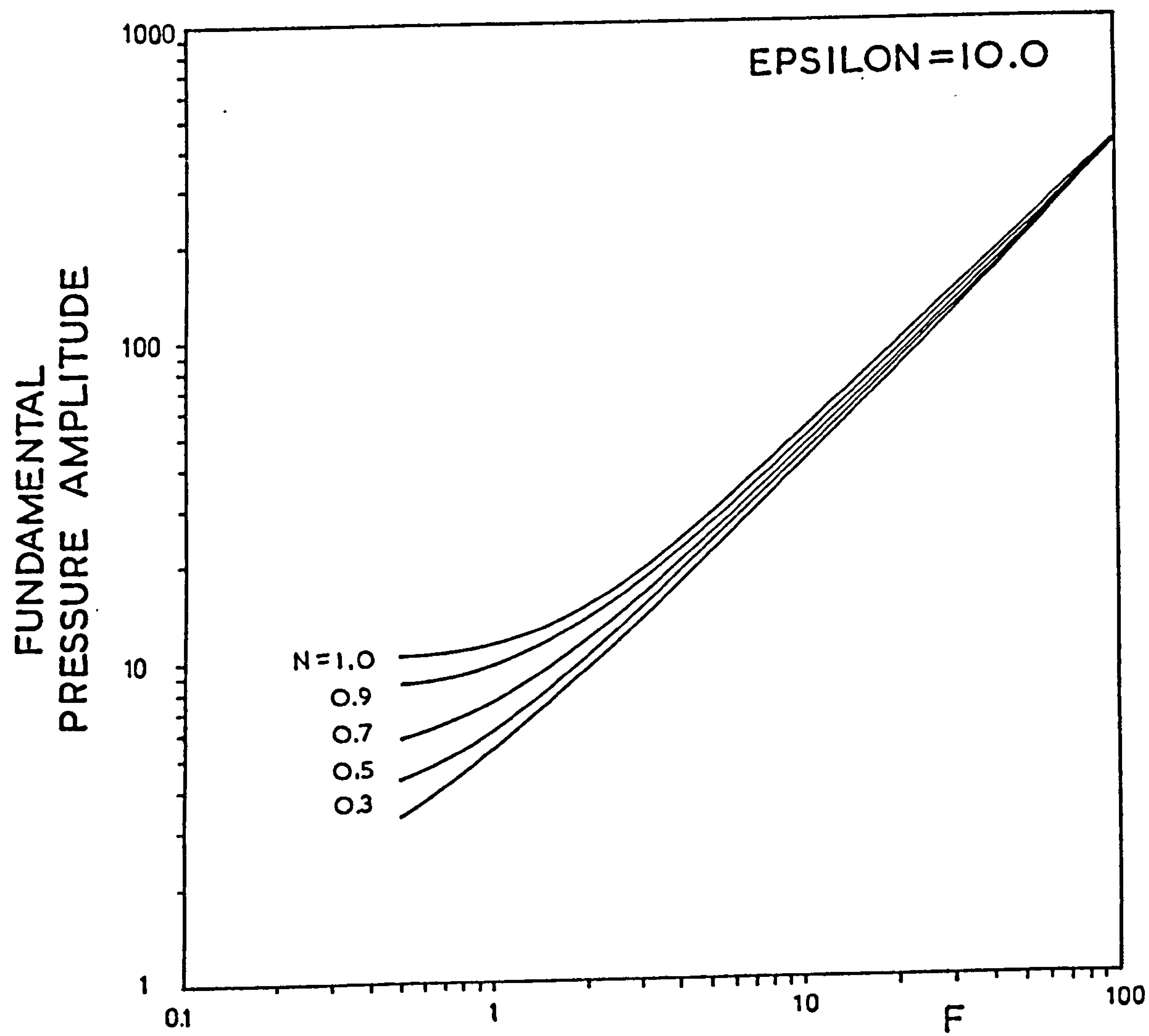
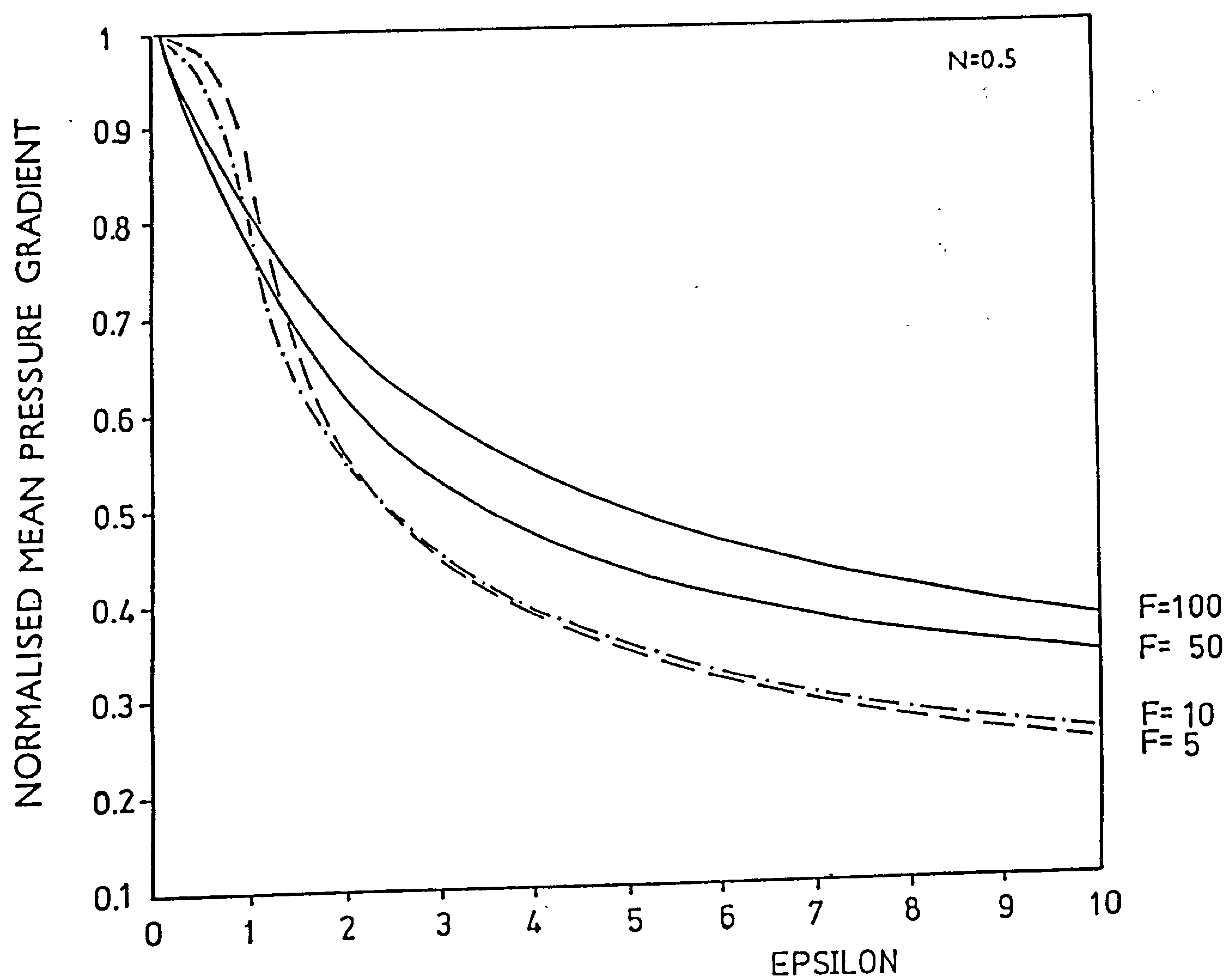

EPSILON=10.0
FUNDAMENTAL
PRESSURE AMPLITUDE
1000
100
10
1
0.1
1
10
100
F
N=1.0
0.9
0.7
0.5
0.3

FIGURE 24

FIGURE 25

THIS FIGURE SHOWS THE VARIATION OF THE NORMALISED MEAN PRESSURE GRADIENT WITH EPSILON WHEN N=0.5 AND F IN THE RANGE 5 TO 100.


N=0.5
NORMALISED MEAN PRESSURE GRADIENT
EPSILON
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
F=100
F= 50
F= 10
F= 5

TABLES 1-3

TABLE 1 TABULATION OF F'(n,d) AGAINST d (REFER FIGURE 8).

TABLE 2 TABULATION OF F'(n) AGAINST NUMBER OF RADIAL ELEMENTS IN THE FINITE ELEMENT MESH. THE RADIUS OF THE MATCHING SURFACE IS 10 TIMES THAT OF THE FIXED SPHERE.

TABLE 3 AS FOR TABLE 2, EXCEPT THAT THE RADIUS OF THE MATCHING SURFACE IS 20 TIMES THAT OF THE FIXED SPHERE. FOR TABLES 2 AND 3 REFER FIGURE 9.

TABLE 1

(MESH PARAMETERS 100*20)

VALUE OF d	F'(n,d)
2	1.76936
10	-0.39778
20	-0.74215
40	-0.96722
60	-1.06053
100	-1.14473

TABLE 2

(MATCH RADIUS OF 10)

MESH PARAMETERS	F'(n)
10*20	-0.71128
20*20	-1.09128
40*20	-1.33409
50*20	-1.3876
70*20	-1.45035
100*20	-1.49911

TABLE 3

(MATCH RADIUS OF 20)

MESH PARAMETERS	F'(n)
10*20	-0.58647
20*20	-0.97847
40*20	-1.24522
50*20	-1.30509
70*20	-1.37626
100*20	-1.43143
140*20	-1.46893

TABLES 4-7

TABULATIONS OF THE NUMERICAL VALUE OF THE NORMALISED MEAN PRESSURE GRADIENT AGAINST F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 4-7 THE VALUE OF EPSILON IS 0.5, 1, 5 AND 10 RESPECTIVELY. REFER FIGURES 13-16.

NORMALISED MEAN PRESSURE GRADIENTS

TABLE 4 (EPSILON=0.5)

N \ F	0.5	1	5	10	50	100
1.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.9	0.99401	0.99396	0.99269	0.99017	0.98076	0.98166
0.7	0.98567	0.98553	0.98178	0.97472	0.93755	0.94064
0.5	0.98220	0.98197	0.97611	0.96557	0.87644	0.89554
0.3	*	0.98514	0.97632	0.96442	0.77136	0.80799

TABLE 5 (EPSILON=1.0)

N \ F	0.5	1	5	10	50	100
1.0	1.00000	1.00001	1.0000	1.00000	1.00000	1.00000
0.9	0.97072	0.97069	0.96321	0.95611	0.94944	0.95459
0.7	0.92544	0.92462	0.89857	0.87487	0.85709	0.87561
0.5	*	0.89767	0.84746	0.80095	0.78187	0.81415
0.3	*	*	0.81482	0.73250	0.74078	0.77536

TABLE 6 (EPSILON=5.0)

N \ F	0.5	1	5	10	50	100
1.0	1.00000	1.00000	0.99999	0.99999	1.00000	0.99998
0.9	0.82467	0.82426	0.81839	0.81667	0.84228	0.86467
0.7	0.55725	0.55552	0.53988	0.53994	0.60622	0.66333
0.5	0.37237	0.36810	0.34611	0.35069	0.43482	0.49442
0.3	0.24461	0.23556	0.21211	0.22543	0.28344	0.31057

TABLE 7 (EPSILON=10.0)

N \ F	0.5	1	5	10	50	100
1.0	1.00000	1.00000	0.99999	1.00000	0.99999	0.99998
0.9	0.76938	0.76895	0.76322	0.76210	0.79214	0.82056
0.7	0.45218	0.45010	0.43745	0.44083	0.51236	0.57189
0.5	0.26191	0.25671	0.24481	0.25444	0.33128	0.37780
0.3	0.14712	0.13725	0.13544	0.14699	0.18570	0.20046

TABLES 8–11

TABULATIONS OF THE THEORETICAL MEAN PRESSURE GRADIENTS AGAINST F FOR N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 8-11 THE VALUES OF EPSILON ARE 0.5, 1, 5 AND 10 RESPECTIVELY. THE MEAN PRESSURE GRADIENTS HAVE BEEN DERIVED USING THE PERTURBATION ANALYSIS. REFER FIGURES 13-16.

NORMALISED MEAN PRESSURE GRADIENTS
DERIVED USING THE PERTURBATION ANALYSIS

TABLE 8 (EPSILON=0.5)

N \ F	1	3	6	8	10	15
1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.9	0.9941	0.9936	0.9921	0.9904	0.9884	0.9811
0.7	0.9860	0.9846	0.9799	0.9750	0.9687	0.9469
0.5	0.9830	0.9808	0.9732	0.9654	0.9553	0.9201
0.3	0.9855	0.9828	0.9736	0.9642	0.9520	0.9097

TABLE 9 (EPSILON=1.0)

N \ F	1	3	6	8	10	15
1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.9	0.9709	0.9678	0.9574	0.9466	0.9328	0.8846
0.7	0.9252	0.9178	0.8927	0.8666	0.8331	0.7169
0.5	0.8990	0.8885	0.8530	0.8162	0.7688	0.6045
0.3	0.8994	0.8878	0.8485	0.8078	0.7554	0.5735

TABLE 10 (EPSILON=5.0)

N \ F	1	3	6	8	10	15
1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.9	0.8244	0.8206	0.8078	0.7944	0.7773	0.7178
0.7	0.5564	0.5419	0.4931	0.4425	0.3775	0.1515
0.5	0.3707	0.3410	0.2408	0.1368	0.0032	
0.3	0.2429	0.1968	0.0412			

TABLE 11 (EPSILON=10.0)

N \ F	1	2	3	4	6	8
1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.9	0.7688	0.7672	0.7647	0.7611	0.7508	0.7364
0.7	0.4495	0.4428	0.4316	0.4160	0.3713	0.3087
0.5	0.2559	0.2400	0.2137	0.1767	0.0723	
0.3	0.1372	0.1090	0.0619			

TABLE 12

TABULATION OF THE THEORETICAL NORMALISED MEAN PRESSURE GRADIENTS (WHEN FLUID INERTIA IS IGNORED) WHEN N IS IN THE RANGE 0.3 TO 1.0 AND EPSILON HAS THE VALUES 0.5, 1, 5, 10, 50 AND 100.

THEORETICAL NORMALISED MEAN PRESSURE GRADIENTS
(FLUID INERTIA IS IGNORED)

TABLE 12

EPS N	0.5	1	5	10
0.3	0.9858	0.9002	0.2537	0.1536
0.4	0.9839	0.8963	0.3090	0.2020
0.5	0.9833	0.9002	0.3768	0.2652
0.6	0.9841	0.9104	0.4593	0.3472
0.7	0.9862	0.9261	0.5592	0.4537
0.8	0.9895	0.9465	0.6798	0.5915
0.9	0.9942	0.9713	0.8251	0.7698
1.0	1.0000	1.0000	1.0000	1.0000

TABLES 13-16

TABULATION OF THE FUNDAMENTAL PHASE ANGLES FOR THE NORMALISED PRESSURE GRADIENTS AGAINST F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 13-16 THE VALUES OF EPSILON ARE 0.5, 1, 5, AND 10 RESPECTIVELY. REFER FIGURES 17-20.

FUNDAMENTAL PHASE ANGLES FOR THE NORMALISED PRESSURE GRADIENTS.

TABLE 13 (EPSILON=0.5)

N \ F	0.5	1	5	10	50	100
1.0	14.555	27.256	64.964	73.240	82.686	84.817
0.9	15.888	29.432	66.578	74.235	83.108	85.146
0.7	19.471	34.959	70 034	76.430	84.023	85.878
0.5	25.244	42.893	73.984	76.729	84.980	86.731
0.3	*	55.070	78.798	82.207	85.663	87.517

TABLE 14 (EPSILON=1.0)

N \ F	0.5	1	5	10	50	100
1.0	14.554	27.255	64.973	73.253	82.686	84.817
0.9	15.577	28.987	66.233	74.033	83.120	85.223
0.7	18.169	33.056	68.784	75.654	84.069	86.093
0.5	*	38.514	71.547	77.486	85.175	87.037
0.3	*	*	75.018	79.840	86.527	88.043

TABLE 15 (EPSILON=5.0)

N \ F	0.5	1	5	10	50	100
1.0	14.555	27.256	64.964	73.240	82.686	84.817
0.9	16.553	30.492	67.320	74.877	83.611	85.607
0.7	21.126	37.361	71.581	77.976	85.422	87.140
0.5	26.331	44.374	75.470	80.901	87.088	88.377
0.3	31.750	51.139	79.126	83.574	88.358	89.150

TABLE 16 (EPSILON=10.0)

N \ F	0.5	1	5	10	50	100
1.0	14.555	27.256	64.964	73.253	82.686	84.812
0.9	17.615	32.138	68.244	75.435	83.839	85.776
0.7	25.230	42.788	73.797	79.351	85.994	87.553
0.5	34.613	53.408	78.478	82.758	87.803	88.812
0.3	44.879	63.042	82.427	85.592	88.953	89.466

TABLES 17-20

TABULATION OF THE FUNDAMENTAL AMPLITUDES OF THE NORMALISED PRESSURE GRADIENTS WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 17-20 THE VALUES OF EPSILON ARE 0.5, 1, 5 AND 10 RESPECTIVELY. REFER FIGURES 21-24.

FUNDAMENTAL AMPLITUDES FOR THE NORMALISED PRESSURE GRADIENTS.

TABLE 17 (EPSILON=0.5)

N \ F	0.5	1	5	10	50	100
1.0	0.51774	0.56745	1.38809	2.50434	10.81144	20.82173
0.9	0.47073	0.52375	1.35566	2.46539	10.74772	20.71905
0.7	0.37723	0.43819	1.28780	2.38013	10.61056	20.50891
0.5	0.28454	0.35592	1.21336	2.28110	10.45428	20.27707
0.3	*	0.27920	1.12640	2.15994	10.28515	20.01692

TABLE 18 (EPSILON=1.0)

N \ F	0.5	1	5	10	50	100
1.0	1.03547	1.13491	2.77587	5.00853	21.62289	41.64345
0.9	0.95753	1.06244	2.72651	4.94801	21.42151	41.29931
0.7	0.80463	0.92106	2.62549	4.82373	21.02242	40.66699
0.5	*	0.77856	2.51421	4.69301	20.63118	40.13605
0.3	*	*	2.38546	4.55598	20.24588	39.72621

TABLE 19 (EPSILON=5.0)

N \ F	0.5	1	5	10	50	100
1.0	5.17735	5.67447	13.8809	25.0434	108.1144	208.2172
0.9	4.52314	5.07056	13.3857	24.3117	106.0478	205.0170
0.7	3.49215	4.14046	12.4938	22.9658	102.4726	200.1439
0.5	2.74466	3.48359	11.7022	21.7795	99.9814	197.6141
0.3	2.19497	3.00483	11.0069	20.7643	98.7472	196.6952

TABLE 20 (EPSILON=10.0)

N \ F	0.5	1	5	10	50	100
1.0	10.3547	11.3489	27.7619	50.0831	216.2289	416.4166
0.9	8.51607	9.66984	26.4788	48.2292	211.2660	408.8970
0.7	5.90482	7.38512	24.2991	44.9993	203.2164	398.4077
0.5	4.29174	6.04360	22.5082	42.3934	198.5377	394.1187
0.3	3.30455	5.22265	21.07021	40.5053	196.8544	392.9862

TABLE 21

TABULATION OF NORMALISED MEAN PRESSURE GRADIENT WITH EPSILON WITH F IN THE RANGE 5 TO 100 AND N EQUAL TO 0.5. REFER FIGURE 25.

NORMALISED MEAN PRESSURE GRADIENTS

TABLE 21 (N=0.5)

F \ EPS	0.1	0.5	1	2	3	4	5	7	10
5	.99808	.97611	.84746	.55845	.45035	.38794	.34611	.29200	.24481
10	.99776	.96557	.80095	.54965	.44856	.38994	.35069	.29946	.25444
50	.99515	.87644	.78187	.61896	.53056	.47442	.43482	.38116	.33128
100	.99237	.89554	.81415	.67739	.59396	.53662	.49442	.43435	.37780

PROGRAM 1

THIS PROGRAM GENERATES THE NODAL POINTS AND THE GEOMETRICAL DATA FOR THE FINITE ELEMENT MESH. THE FILE FOLLOWING THE PROGRAM REPRESENTS A TYPICAL OUTPUT FILE CONTAINING THIS DATA.

```
                  C
C
C                         THIS PROGRAM GENERATES
C                     THE NODAL POINTS AND GEOMETRICAL DATA
C                        FOR THE FINITE ELEMENT MESH
C
C

      INTEGER I,J,TOTNOD, V1,N,M,NX,NY,ISTAR,IFINI,
     *        W1,W2,W3,W4,W5,NELX,DOFNOD,DIMEN,ELTYP,NEL,NODEL
      DOUBLE PRECISION L,V2,V3,D,NXD,NYD
      DIMENSION V1(4000),V2(4000),V3(4000),
     *        W1(4000),W2(6400),W3(6400),W4(6400),W5(6400)
      COMMON/A/W1,W2
      COMMON/B/W3,W4
      COMMON/C/W5
      COMMON/D/V1
      COMMON/E/V2
      COMMON/F/V3
      DATA NX/101/,NY/21/,TOTNOD/2121/,DIMEN/2/,
     *     NEL/2000/,DOFNOD/1/,ELTYP/1/,NODEL/4/,NOUT/6/
      DO 10 I=1,TOTNOD
      V1(I)=I
   10 CONTINUE
C
      N=1
      L=0.0D0
      DO 20 J=1,NX
      ISTAR=N
      IFINI=N+NY-1
      DO 30 I=ISTAR,IFINI
      V2(I)=L
   30 CONTINUE
      N=N+NY
      NXD=NX*1.0D0
C     L=L+0.6931472/(NXD-1)
C     L=L+2.9957323/(NXD-1)
C     L=L+3.4011974/(NXD-1)
C     L=L+3.6888795/(NXD-1)
      L=L+2.30258/(NXD-1)
C     L=L+4.09434/(NXD-1)
   20 CONTINUE
C
      N=1
      DO 40 I=1,NX
      M=0
      JSTAR=N
      JFINI=N+NY-1
      DO 50 J=JSTAR,JFINI
      NYD=NY*1.0D0
      D=-2*M/(NYD-1)+1
      V3(J)=D
      M=M+1
   50 CONTINUE
      N=N+NY
   40 CONTINUE
      WRITE(NOUT,8010)TOTNOD,DIMEN
```

```
      DO 60 I=1,TOTNOD
      WRITE(NOUT,8020)V1(I),V2(I),V3(I)
   60 CONTINUE
C
      DO 70 I=1,NEL
      W1(I)=I
   70 CONTINUE
C
      NELX=NX-1
      M=1
      N=0
      DO 80 I=1,NELX
      JSTAR=N+1
      JFINI=N+NY-1
      DO 90 J=JSTAR,JFINI
      W2(J)=M
      M=M+1
      N=N+1
   90 CONTINUE
      M=M+1
   80 CONTINUE
C
      DO 100 I=1,NEL
      W3(I)=W2(I)+1
      W4(I)=W3(I)+NY
      W5(I)=W2(I)+NY
  100 CONTINUE
C
      WRITE(NOUT,8010)ELTYP,NEL,NODEL
      DO 110 I=1,NEL
      WRITE(NOUT,8010)W1(I),W2(I),W3(I),W4(I),W5(I)
  110 CONTINUE
C
      WRITE(NOUT,8010)DOFNOD
C
      STOP
 8010 FORMAT(16I5)
 8020 FORMAT(I5,6D12.5)
      END
```

THE DATA FILE FOR
A FINITE ELEMENT MESH

TOTAL NUMBER OF NODES IN FE MESH
DIMENSIONALITY OF PROBLEM

16 2

NODAL GEOMETRY

GLOBAL NODE NUMBER AND
COORDINATES OF NODE

```
 1 0.00000D+00 0.10000D+01
 2 0.00000D+00 0.33333D+00
 3 0.00000D+00-0.33333D+00
 4 0.00000D+00-0.10000D+01
 5 0.76753D+00 0.10000D+01
 6 0.76753D+00 0.33333D+00
 7 0.76753D+00-0.33333D+00
 8 0.76753D+00-0.10000D+01
 9 0.15351D+01 0.10000D+01
10 0.15351D+01 0.33333D+00
11 0.15351D+01-0.33333D+00
12 0.15351D+01-0.10000D+01
13 0.23026D+01 0.10000D+01
14 0.23026D+01 0.33333D+00
15 0.23026D+01-0.33333D+00
16 0.23026D+01-0.10000D+01
```

ELEMENT GEOMETRY

ELEMENT TYPE,NUMBER OF ELEMENTS
AND NUMBER OF NODES PER ELEMENT

1 9 4

NUMBER OF ELEMENT AND THE GLOBAL
NUMBERING OF IT'S VERTEX NODES

1	1	2	6	5
2	2	3	7	6
3	3	4	8	7
4	5	6	10	9
5	6	7	11	10
6	7	8	12	11
7	9	10	14	13
8	10	11	15	14
9	11	12	16	15

DEGREES OF
FREEDOM PER NODE

1

PROGRAM 2

THIS PROGRAM CALCULATES THE DRAG CORRECTION FACTOR FOR A SPHERE FALLING THROUGH AN INFINITE EXPANSE OF SLIGHTLY POWER LAW FLUID.

```
C
C
C
C         THIS PROGRAM CALCULATES THE DRAG CORRECTION
C         FACTOR FOR A SPHERE FALLING SLOWLY THROUGH
C         AN INFINITE EXPANSE OF POWER LAW FLUID.
C
C
      INTEGER*4   BDCND, BTYPES, DIF, DIMEN, DOFEL,DIAG,ILO,ISTIF,
     * DOFNOD, ELNUM, ELTOP, ELTYP, HBAND, I, IABSS,CI,
     * ICOORD,ILOAD,ICNT,LL,IPFINI,SP,TCI,
     * IDTPD, IELK, IELTOP, IFUN, IGDER,IPTR,IPSTAR,
     * IGEOM,IJAC,IJACIN,ILDER,INF,IP,
     * IQUAD,ISTEER,ITEST,IWGHT,REG,NN,KK1,KSP,SN,
     * J,JABSS,JCOORD,LAST,IRN,ILOCAL,II,POD,KOUNT,
     * JDTPD,JELK,JELTOP,JGDER,JGEOM,JJAC,
     * JJACIN, JLDER, JNF, JP, K,  NELE,ILV,ILEC,
     * NF, NIN, NODEL, NODNUM, NOUT, NQP,
     * STEER,TOTDOF,TOTELS,TOTNOD,CCC,COUNT1,COUNT2,COUNT3,
     * BCD1,BCD2,IMID,JMID,
     * IF,JF,COUNT,ILN,IBEV,IRH,
     * ICP,ICCP,INTRA,INTRB
      DOUBLE PRECISION ABSS,DENOMA,SUM,STIF,RRQ,SINARG,VN1,N,
     * COORD, DET, DTPD, ELK, ETA, FUN, GDER,DENOMB,TY,
     * GDERT, GEOM, JAC, JACIN, LDER, P, PD, QUOT,DIV,
     * SCALE, WGHT, X, X1, XX, Y, YY,LOAD,LEC,LV,LO,
     * WGA,H1,H2, MID,F,R,RQ,TQ,MGDER(2,4),IW,SLOPE,ARGLOG,
     * FMD,PI,LN,BEV,RH, U, OMVAA,SLOP,FN,RSTAR,GSTAR,RMARG,
     * OMVAB,S(5), GU,TTT,RRR,G1,G2,G3,G4,G5,G6,G7,G8,IPOINT,
     * S1,S2,S3,S4,S5,S6,A1,B1,P1,P2,P3,P4,P5,P6,BETAP,JTC4,
     * C1,D1,COEFF,XE(3000,4,4),YE(4,4),ALPHA,
     * DGU(40),DANEW(40),DRGU(40),DRANEW(40),LMULT1,LMULT2
      LOGICAL FIRST
      DIMENSION ABSS(2,9),ILOCAL(9),
     * DTPD(4,4),
     * ELK(4,4), FUN(4), GDER(2,4), GDERT(4,2), GEOM(4,2),
     * JAC(2,2), JACIN(2,2), LDER(2,4), P(2,2), PD(2,4),
     * STEER(4), WGHT(9),MID(2,2),F(4,2),R(2),LN(4),LV(4),
     * FMD(4,4),BEV(4),LEC(4),LO(6561)
C     PROBELM SIZE DEPENDENT ARRAYS
      DIMENSION LOAD(6561), GU(6561), COORD(6561,2),
     * ELTOP(6561,6), NF(6561,1), COEFF(3000,9),
     * ANEW(6561),RH(6561), RSTAR(6561),GSTAR(6561),
     * OMVAA(81), OMVAB(81), INTRA(81),
     * INTRB(81),
     * IW(81), BCD1(5,81), BCD2(5,81),
     * BDCND(2,2,81),REG(6562),IRN(65620),DIAG(6561),
     * STIF(65620)
      COMMON/A/ELTOP
      COMMON/B/DIAG
      COMMON/C/COORD
      COMMON/D/RH,ANEW,NF
      COMMON/E/ABSS,BCD1,BCD2,BDCND,BEV,DTPD,
     * ELK,F,FMD,FUN,GDER,
     * GDERT,GEOM,GU,INTRA,INTRB,IW
      COMMON/F/JAC,JACIN,LDER,LN,LOAD,MGDER,MID,
     * OMVAA,OMVAB,P,PD,R,S,STEER,
```

```
     * W1,W2,WGHT,REG,IRN
      COMMON/G/STIF,COEFF
      COMMON/H/GSTAR,RSTAR
      COMMON/I/XE
      COMMON/J/YE
      DATA IABSS /2/,ISTIF/65620/,
     * IDTPD /4/,IELK /4/,
     * IFUN /4/, IGDER /2/,IGEOM /4/, IJAC /2/,
     * IJACIN /2/, ILDER /2/, IP /2/,ISTEER /4/,
     * IWGHT /9/, JABSS /9/,JCOORD /2/, ILV/4/, ILEC/4/,
     * JDTPD /4/, JELK /4/, JGDER /4/,JGEOM /2/,
     * JJAC /2/, JJACIN /2/, JLDER /4/, JNF /1/, JP /2/,
     * SCALE /1.0D+25/,ILOAD/6561/,
     * IMID /2/,JMID/2/,IF/4/,JF/2/, ILO/6561/
C     PROBELM SIZE DEPENDENT DATA STATEMENTS
      DATA ICOORD/6561/,IELTOP/6561/,INF/6561/,
     * JELTOP/6/, LMULT1/0.5/, LMULT2/0.5/,
     * IRH/6561/,IBEV/4/,ILN/4/,
     * RR/10.0/, BETA/-1.5/, COUNT/21/,
     * ALPHA/0.0/
      DATA NIN /5/, NOUT /6/
C     WGA(XX,YY) = 1.0*((DSIN(YY)**2)/34.0)*(90*XX**2-372/XX)
      WGA(XX,YY)=-0.5*(-2*BB/XX+10*DD*XX**2)*DSIN(YY)**2
      H1(XX,YY)=0.0000D0
C     H2(XX,YY)=-(DSIN(YY)**2*2.0)
      H2(XX,YY)= -(DSIN(YY)**2*0.5*RR**2)
C     H3(XX,YY)=-0.5*(BETA*RR+RR**2)*DSIN(YY)**2
      H3(XX,YY)=(3*(ALPHA/RR+RR*BETA+2*RR*ALOG(RR))/4.0)*DSIN(YY)**2
C     H3(XX,YY)=-(DSIN(YY)**2*RR**2)/2.0
C     H3(XX,YY)=0.0000
C     SET ITEST FOR FULL CHECKING
      ITEST = 0
      PI=4.0D0*DATAN(1.0D0)
C
      AA=2*RR**3*(1+RR+RR**2)/(RR-1)**3/
     *             (4+7*RR+4*RR**2)
      BB=-6*RR*(1+RR+RR**2+RR**3+RR**4)/
     *           (RR-1)**3/(4+7*RR+4*RR**2)
      CC=RR*(9+9*RR+4*RR**2+4*RR**3+4*RR**4)/
     *        (RR-1)**3/(4+7*RR+4*RR**2)
      DD=-3*RR*(1+RR)/(RR-1)**3/(4+7*RR+4*RR**2)
      FN=24*RR*(1+RR+RR**2+RR**3+RR**4)/
     *           ((RR-1)**3*(4+7*RR+4*RR**2)*1.0)
      ARGLOG=2*AA+2*CC+12*DD
      DIV=4*(5*AA+2*CC)/3.0
      SLOPE=2*(0.75*(2*DLOG(ARGLOG)/3.0-5/9.0)
     *        +(4*AA+2*CC+4*DD)/DIV)
      AA=0.5
      BB=-1.5
      CC=1.0
      DD=0.0
      ARGLOG=2*AA+2*CC+12*DD
      DIV=4*(5*AA+2*CC)/3.0
      SLOPE=2*(0.75*(2*DLOG(ARGLOG)/3.0-5/9.0)
     *        +(4*AA+2*CC+4*DD)/DIV)
C
C      INPUT OF DATA FROM FINITE ELEMENT MESH
```

```
C       (1) NODAL GEOMETRY
C
        OPEN(FILE='LLDATA',UNIT=7,STATUS='UNKNOWN')
        READ(NIN,8010) TOTNOD,DIMEN
        WRITE(7,8010)TOTNOD
        DO 1010 I=1,TOTNOD
        READ(NIN,8020) NODNUM,(COORD(NODNUM,J),J=1,DIMEN)
 1010   CONTINUE
C
C       (2) ELEMENT TOPOLOGY
C
        READ (NIN,8010) ELTYP, TOTELS, NODEL
        WRITE(7,8010)TOTELS
        DO 1020 I=1,TOTELS
        READ (NIN,8010) ELNUM, (ELTOP(ELNUM,J+2),J=1,NODEL)
        ELTOP(ELNUM,1) = ELTYP
        ELTOP(ELNUM,2) = NODEL
 1020   CONTINUE
C
        READ (NIN,8010) DOFNOD
        BTYPES=1
        DO 2324 I=1,COUNT
        J=I+3
        BCD1(1,J)=I
        BCD2(1,J)=I+TOTNOD-COUNT
 2324   CONTINUE
        DO 1075 J=1,COUNT
        JJ=J+3
        BDCND(1,1,JJ)=BCD1(1,JJ)
        BDCND(2,2,JJ)=BCD2(1,JJ)
 1075   CONTINUE
        CCC=COUNT+3
        K=COUNT+3
        DO 2071 I=1,COUNT
        INTRA(I)=BCD1(1,I+3)+COUNT
        INTRB(I)=BCD2(1,I+3)-COUNT
 2071   CONTINUE
        WRITE(1,8010) (INTRA(I),I=1,COUNT)
        WRITE(1,8010) (INTRB(I),I=1,COUNT)
        TOTDOF = 0
        DO 1050 I=1,TOTNOD
        DO 1040 J=1,DOFNOD
        TOTDOF = TOTDOF + 1
        NF(I,J) = TOTDOF
 1040   CONTINUE
 1050   CONTINUE
C
C       ASSEMBLY OF THE MODIFIED STIFFNESS MATRIX
C:      (ONLY NON-ZERO ELEMENTS ARE INCLUDED)
C
        IPTR=1
        CI=1
        WRITE(1,8023)
        JJF=1
        ICOUN1=COUNT-1
        ITOTC2=TOTELS-COUNT+2
        ITNC=TOTNOD-COUNT
```

```
      NNN=0
      DO 23 I=1,TOTNOD
      TCI=CI-I
      IF(TCI.GT.0.00001)GO TO 231
      CI=CI+50
      WRITE(1,8010) I
 231  IPSTAR=IPTR
      REG(I)=IPTR
      IRN(IPTR)=I
      IPTR=IPTR+1
      IF(I.LE.COUNT)GO TO 6765
      IF(I.GT.ITNC)GO TO 7765
      ITT=I-1-(COUNT+NNN*COUNT)
      IF(ITT.NE.0)GO TO 8765
      JJF=1+NNN*(COUNT-1)
      JJL=JJF+2*(COUNT-1)-1
      NNN=NNN+1
8765  DO 24 J=JJF,JJL
      CALL DIRECT(J,ELTOP,IELTOP,JELTOP,NF,INF,JNF,
     *            DOFNOD,STEER,ISTEER,ITEST)
      DO 25 K=1,4
      IF(STEER(K).EQ.I)GO TO 15
  25  CONTINUE
      GO TO 24
  15  CONTINUE
      DO 30 K=1,4
      IPFINI=IPTR-1
      DO 26 L=IPSTAR,IPFINI
      IF(IRN(L).EQ.STEER(K))GO TO 30
  26  CONTINUE
      IRN(IPTR)=STEER(K)
      IPTR=IPTR+1
  30  CONTINUE
  24  CONTINUE
      GO TO 7799
7765  DO 224 J=ITOTC2,TOTELS
      CALL DIRECT(J,ELTOP,IELTOP,JELTOP,NF,INF,JNF,
     *            DOFNOD,STEER,ISTEER,ITEST)
      DO 225 K=1,4
      IF(STEER(K).EQ.I)GO TO 215
 225  CONTINUE
      GO TO 224
 215  CONTINUE
      DO 230 K=1,4
      IPFINI=IPTR-1
      DO 226 L=IPSTAR,IPFINI
      IF(IRN(L).EQ.STEER(K))GO TO 230
 226  CONTINUE
      IRN(IPTR)=STEER(K)
      IPTR=IPTR+1
 230  CONTINUE
 224  CONTINUE
      GO TO 7799
6765  DO 1224 J=1,ICOUN1
      CALL DIRECT(J,ELTOP,IELTOP,JELTOP,NF,INF,JNF,
     *            DOFNOD,STEER,ISTEER,ITEST)
      DO 325 K=1,4
```

```
      IF(STEER(K).EQ.1)GO TO 315
  325 CONTINUE
      GO TO 1224
  315 CONTINUE
      DO 330 K=1,4
      IPFINI=IPTR-1
      DO 326 L=IPSTAR,IPFINI
      IF(IRN(L).EQ.STEER(K))GO TO 330
  326 CONTINUE
      IRN(IPTR)=STEER(K)
      IPTR=IPTR+1
  330 CONTINUE
 1224 CONTINUE
 7799 LAST=IPSTAR+8
   23 CONTINUE
      REG(TOTNOD+1)=IPTR
      ITOT1=TOTNOD+1
      DO 474 I=1,ITOT1
      WRITE(7,8010)REG(I)
  474 CONTINUE
      CI=1
      WRITE(1,8037)
      DO 36 I=1,TOTNOD
      TCI=CI-I
      IF(TCI.GT.0.00001)GO TO 361
      CI=CI+400
      WRITE(1,8010) I
  361 IP1=REG(I)
      IP2=REG(I+1)-1
      K=0
      DO 31 J=IP1,IP2
      K=K+1
      ILOCAL(K)=IRN(J)
   31 CONTINUE
      KK1=K-1
      DO 32 SP=1,KK1
      M=0
      KSP=K-SP
      DO 33 NN=1,KSP
      IF(ILOCAL(NN).LE.ILOCAL(NN+1)) GO TO 33
      X=ILOCAL(NN)
      ILOCAL(NN)=ILOCAL(NN+1)
      ILOCAL(NN+1)=X
      M=1
   33 CONTINUE
      IF(M.EQ.0)GO TO 34
   32 CONTINUE
   34 K=0
      DO 35 J=IP1,IP2
      K=K+1
      IRN(J)=ILOCAL(K)
   35 CONTINUE
   36 CONTINUE
      IPEND=J-1
      IPEND1=IPEND+1
      IPENDT=IPEND+TOTNOD
      WRITE(7,8010) IPEND
```

```fortran
      WRITE(7,8010) IPEND1
      WRITE(7,8010) IPENDT
C     CALCULATION OF SEMI-BANDWIDTH
      FIRST = .TRUE.
      DIF=COUNT+1
      HBAND = DIF + 1
      WRITE(1,8038)
      WRITE(1,8010) HBAND
      DOFEL=NODEL*DOFNOD
      CALL VECNUL(STIF,ISTIF,ISTIF,ITEST)
      CALL MATNUL(MID,IMID,JMID,DIMEN,DIMEN,ITEST)
      CALL MATNUL(P,IP,JP,DIMEN,DIMEN,ITEST)
      CALL MATNUL(F,IF,JF,DOFEL,DIMEN,ITEST)
      CALL QQUA4(WGHT, IWGHT, ABSS, IABSS,JABSS, NQP, ITEST)
      CI=1
      WRITE(1,8043)
      DO 1100 NELE=1,TOTELS
      TCI=CI-NELE
      IF(TCI.GT.0.00001)GO TO 1101
      CI=CI+200
      WRITE(1,8010)NELE
 1101 KCOUNT=0
      CALL DIRECT(NELE,ELTOP,IELTOP,JELTOP,NF,INF,JNF,
     *            DOFNOD,STEER,ISTEER,ITEST)
      CALL ELGEOM(NELE, ELTOP, IELTOP, JELTOP, COORD, ICOORD,
     *     JCOORD, GEOM, IGEOM, JGEOM, DIMEN, ITEST)
      CALL MATNUL(ELK, IELK, JELK, DOFEL, DOFEL, ITEST)
      CALL VECNUL(BEV,IBEV,DOFEL,ITEST)
      DO 1090 IQUAD=1,NQP
      XI = ABSS(1,IQUAD)
      ETA = ABSS(2,IQUAD)
      CALL QUAM4(FUN, IFUN, LDER, ILDER, JLDER, XI, ETA, ITEST)
      CALL MATMUL(LDER, ILDER, JLDER, GEOM, IGEOM, JGEOM, JAC,
     *     IJAC, JJAC, DIMEN, NODEL, DIMEN, ITEST)
      CALL MATINV(JAC, IJAC, JJAC, JACIN, IJACIN, JJACIN, DIMEN,
     *     DET, ITEST)
      CALL MATMUL(JACIN, IJACIN, JJACIN, LDER, ILDER, JLDER, GDER,
     *     IGDER, JGDER, DIMEN, DIMEN, NODEL, ITEST)
      CALL SCAPRD(FUN,IFUN,GEOM(1,1),IGEOM,NODEL,RQ,ITEST)
      CALL SCAPRD(FUN,IFUN,GEOM(1,2),IGEOM,NODEL,TQ,ITEST)
      DO 1212 I=1,4
      DO 1313 J=1,4
      DTPD(I,J)=-GDER(1,I)*GDER(1,J)-
     *GDER(1,J)*FUN(I)-
     *(1-TQ**2)*GDER(2,I)*GDER(2,J)+
     *2*TQ*GDER(2,J)*FUN(I)
 1313 CONTINUE
 1212 CONTINUE
      QUOT = DET*WGHT(IQUAD)
      DO 1080 I=1,DOFEL
      DO 1070 J=1,DOFEL
      DTPD(I,J) = DTPD(I,J)*QUOT
 1070 CONTINUE
 1080 CONTINUE
      CALL MATADD(ELK, IELK, JELK, DTPD, IDTPD, JDTPD, DOFEL,
     *     DOFEL, ITEST)
 1090 CONTINUE
```

```
      DO 51 H=1,TOTNOD
      IF(KCOUNT.EQ.NODEL)GO TO 1100
      DO 52 I=1,4
      IF(H.EQ.STEER(I))GO TO 56
   52 CONTINUE
      GO TO 51
   56 KCOUNT=KCOUNT+1
      IP1=REG(H)
      IP2=REG(H+1)-1
      DO 53 K=1,4
      DO 54 L=IP1,IP2
      IF(IRN(L).EQ.STEER(K))STIF(L)=STIF(L)+ELK(I,K)
   54 CONTINUE
   53 CONTINUE
   51 CONTINUE
 1100 CONTINUE
      DO 774 I=1,IPEND
      WRITE(7,8011)STIF(I)
  774 CONTINUE
      DO 779 I=1,IPEND
      WRITE(7,8010)IRN(I)
  779 CONTINUE
      DO 58 I=1,TOTNOD
      IP1=REG(I)
      IP2=REG(I+1)-1
   58 CONTINUE
      DO 59 I=1,TOTNOD
      IP1=REG(I)
      IP2=REG(I+1)-1
      KOUNT=1
      DO 60 L=IP1,IP2
      IF(IRN(L).LT.I)GO TO 61
      IF(IRN(L).EQ.I)GO TO 62
   61 KOUNT=KOUNT+1
   60 CONTINUE
   62 DIAG(I)=KOUNT
   59 CONTINUE
      ICP=TOTNOD-2*COUNT+1
      ICCP=TOTNOD-COUNT
      DO 1115 I=1,2
      DO 1120 J=1,COUNT
      K=BDCND(I,I,J+3)
      STIF(POD)=STIF(POD)*SCALE
      POD=REG(K)+DIAG(K)-1
 1120 CONTINUE
 1115 CONTINUE
      DO 473 I=1,TOTNOD
      WRITE(7,8010)DIAG(I)
  473 CONTINUE
      IPA=COUNT+1
      DO 1253 K=IPA,ICP,COUNT
      POD=REG(K)+DIAG(K)-1
      STIF(POD)=STIF(POD)*SCALE
 1253 CONTINUE
      IPB=2*COUNT
      DO 1252 K=IPB,ICCP,COUNT
      POD=REG(K)+DIAG(K)-1
```

```
      STIF(POD)=STIF(POD)*SCALE
 1252 CONTINUE
      DO 68 I=1,TOTNOD
      IP1=REG(I)
      IP2=REG(I+1)-1
   68 CONTINUE
C
C      ASSEMBLY OF THE LOAD VECTOR
C      FOR RHS=F(R,THETA)
C
      CALL VECNUL(LO,ILO,ILO,ITEST)
      DO 2100 NELE=1,TOTELS
      CALL ELGEOM(NELE,ELTOP,IELTOP,JELTOP,COORD,ICOORD,
     *        JCOORD,GEOM,IGEOM,JGEOM,DIMEN,ITEST)
      CALL  VECNUL(LV,ILV,DOFEL,ITEST)
      CALL DIRECT(NELE,ELTOP,IELTOP,JELTOP,NF,INF,JNF,
     *     DOFNOD,STEER,ISTEER,ITEST)
      DO 2091  IQUAD=1,NQP
      XI=ABSS(1,IQUAD)
      ETA=ABSS(2,IQUAD)
      CALL QUAM4(FUN,IFUN,LDER,ILDER,JLDER,XI,ETA,ITEST)
      CALL  MATMUL(LDER,ILDER,JLDER,GEOM,IGEOM,JGEOM,JAC,
     *      IJAC,JJAC,DIMEN,NODEL,DIMEN,ITEST)
      DET=JAC(1,1)*JAC(2,2)-JAC(1,2)*JAC(2,1)
      CALL SCAPRD(FUN,IFUN,GEOM(1,1),IGEOM,NODEL,RQ,ITEST)
      CALL SCAPRD(FUN,IFUN,GEOM(1,2),IGEOM,NODEL,TQ,ITEST)
      QUOT=DET*WGHT(IQUAD)
      RRR=DEXP(RQ)
      IF(TQ.EQ.0)GO TO 45
      TY=DSQRT((1-TQ**2)/TQ**2)
      TTT=DATAN(TY)
      GO TO 46
   45 TTT=PI/2.0
   46 G1=2*DCOS(TTT)*(-3*AA/RRR**4-BB/RRR**2+2*DD*RRR)
      G2=-G1/2
      G3=G2
      G4=-3*DSIN(TTT)*(AA/RRR**4+DD*RRR)
      G5=2*DCOS(TTT)*(12*AA/RRR**5+2*BB/RRR**3+2*DD)
      G6=-2*DSIN(TTT)*(-3*AA/RRR**4-BB/RRR**2+2*DD*RRR)
      G7=-3*DSIN(TTT)*(-4*AA/RRR**5+DD)
      G8=-3*DCOS(TTT)*(AA/RRR**4+DD*RRR)
      S1=2*DCOS(TTT)*(-60*AA/RRR**6-6*BB/RRR**4)
      S2=-2*DSIN(TTT)*(12*AA/RRR**5+2*BB/RRR**3+2*DD)
      S3=-G1
      S4=-3*DSIN(TTT)*(20*AA/RRR**6)
      S5=-3*DCOS(TTT)*(-4*AA/RRR**5+DD)
      S6=-G4
      A1=-DSIN(TTT)*(10*DD*RRR+BB/RRR**2)
      B1=2*(5*DD-BB/RRR**3)*DCOS(TTT)
      P1=3*G1**2/2.0+2*G4**2
      P2=3*G1*G5+4*G4*G7
      P3=3*G1*G6+4*G4*G8
      P4=3*G1*S1+3*G5**2+4*G4*S4+4*G7**2
      P5=3*G1*S2+3*G6*G5+4*G4*S5+4*G8*G7
      P6=3*G1*S3+3*G6**2+4*G4*S6+4*G8**2
      C1=G1*(P1*P5-P2*P3)/(P1**2)+G6*P2/P1
      C1=C1+(P6/P1-(P3/P1)**2)*G4/RRR+G8*P3/(RRR*P1)
```

```
      D1=(G4+RRR*G7)*P2/P1+RRR*G4*(P1*P4-P2**2)/(P1**2)
      D1=D1-G5*P3/(2*P1)-(G1/2.0)*(P1*P5-P3*P2)/(P1**2)
      COEFF(NELE,IQUAD)=DSIN(TTT)/2.0*(C1-D1+P3*B1/P1
     *                                  -P2*A1/P1)
      VN1=N-1
      DO 7086 K=1,DOFEL
      RRQ=2.0*RQ
      XE(NELE,IQUAD,K)=FUN(K)*QUOT*DEXP(RRQ)
      YE(IQUAD,K)=FUN(K)
      LEC(K)=FUN(K)*COEFF(NELE,IQUAD)*QUOT*DEXP(RRQ)
 7086 CONTINUE
      CALL VECADD(LV,ILV,LEC,ILEC,DOFEL,ITEST)
 2091 CONTINUE
      DO 2222 L=1,DOFEL
      SN=STEER(L)
      LO(SN)=LO(SN)+LV(L)
 2222 CONTINUE
 2100 CONTINUE
      DO 776 I=1,TOTNOD
      WRITE(7,8011)LO(I)
  776 CONTINUE
      DO 512 I=1,4
      DO 513 J=1,4
      WRITE(7,8011)YE(I,J)
  513 CONTINUE
  512 CONTINUE
      DO 514 I=1,TOTELS
      DO 515 J=1,4
      DO 516 K=1,4
      WRITE(7,8011)XE(I,J,K)
  516 CONTINUE
  515 CONTINUE
  514 CONTINUE
C
      COUNT1=COUNT+1
      COUNT2=2*COUNT+1
      COUNT3=TOTNOD-COUNT
      DENOMA=COORD(COUNT1,1)
      DENOMJ=DEXP(COORD(COUNT2,1))-DEXP(COORD(1,1))
      DENOMB=DEXP(COORD(TOTNOD,1))-DEXP(COORD(COUNT3,1))
C
C     CALCULATION OF THE NEWTONIAN PSI FIELD
C
      DO  1255  I=1,TOTNOD
      IF(COORD(I,2).EQ.0)GO TO 14
      TY=DSQRT((1-COORD(I,2)**2)/COORD(I,2)**2)
      SINARG=DATAN(TY)
      GO TO 17
   14 SINARG=PI/2.0
   17 U=DSIN(SINARG)
      XY=DEXP(COORD(I,1))
      ANEW(I)=-0.5*(AA/XY+BB*XY+
     *              CC*XY**2+DD*XY**4)*U**2
      RH(I)=ANEW(I)
      RSTAR(I)=ANEW(I)
 1255 CONTINUE
 2057 FORMAT(51H ---------------------------------------------------)
```

```
C
C     CALCULATION OF THE NEWTONIAN OMEGA FIELD
C
      DO 719  I=1,TOTNOD
      IF(COORD(I,2).EQ.0)GO TO 9
      TY=DSQRT((1-COORD(I,2)**2)/COORD(I,2)**2)
      SINARG=DATAN(TY)
      GO TO 10
    9 SINARG=PI/2.0
   10 GU(I)=WGA(DEXP(COORD(I,1)),SINARG)
      GSTAR(I)=GU(I)
  719 CONTINUE
C
      IPOINT=0.0000D0
      JTC=0
      ICNT=0
      WRITE(1,8050)
      WRITE(NOUT,8050)
      WRITE(1,787)
      WRITE(1,8036)DENOMA,DENOMB
  787 FORMAT(37H                       INTO ITERATIVE LOOP //)
      CLOSE(7)
      IDGU=COUNT-ICOUN1/2.0
C
C     MAIN ITERATIVE LOOP
C     ...................
C     CALCULATION OF THE BOUNDARY OMEGA VALUES
C
  725 DO 2073  I=1,COUNT
      JC1=I+1*COUNT
      OMVAA(I)=2*RH(JC1)/DENOMA/DENOMA
C     OMVAA(I)=-1.5*DSIN(SINARG)**2
C     SINARG=DACOS(COORD(INTRB(I),2))
C     OMVAB(I)=0.1*BETA*DSIN(SINARG)**2
      IF(COORD(INTRB(I),2).EQ.0)GO TO 47
      TY=DSQRT((1-COORD(INTRB(I),2)**2)/COORD(INTRB(I),2)**2)
      SINARG=DATAN(TY)
      GO TO 48
   47 SINARG=PI/2.0
   48 OMVAB(I)=(-3*(BETA-1+2*ALOG(RR))*DSIN(SINARG)**2)/(2.0*RR)
C     OMVAB(I)=2*RH(INTRB(I))/DENOMB/DENOMB
C    *           (RR-DENOMB)**2*DSIN(SINARG)**2/DENOMB
C    *        /DENOMB
      DRGU(I)=(-3)*(3-BETA-2*ALOG(RR))*DSIN(SINARG)**2/(2.0*RR**2)
      DRANEW(I)=(3/4.0)*(-ALPHA/RR**2+BETA+2+2*ALOG(RR))*
     *            DSIN(SINARG)**2
 2073 CONTINUE
C
      CALL VECNUL(LOAD,ILOAD,ILOAD,ITEST)
      DO 2074  I=1,COUNT
      J=I+3
      LOAD(BCD1(1,J))=OMVAA(I)
      LOAD(BCD2(1,J))=OMVAB(I)
 2074 CONTINUE
      LL=TOTNOD-COUNT+1
      DO 1111  I=1,TOTNOD
      RH(I)=LO(I)
```

```
 1111 CONTINUE
C
C     APPLICATION OF DIRICHLET BOUNDARY
C     CONDITIONS FOR OMEGA
C
      DO 4115 I=1,2
      DO 4120 J=1,COUNT
      K = BDCND(I,I,J+3)
      X = COORD(K,1)
      GO TO(4125,4135),I
 4125 POD=REG(K)+DIAG(K)-1
      RH(K)=STIF(POD)*LOAD(K)
      GO TO 4120
 4135 POD=REG(K)+DIAG(K)-1
      RH(K)=STIF(POD)*LOAD(K)
 4120 CONTINUE
 4115 CONTINUE
      ICCP=COUNT+COUNT*(COUNT-1)
      DO 4253 K=1,ICP,COUNT
      RH(K)=0.0000D0
 4253 CONTINUE
      DO  4252 K=COUNT,ICCP,COUNT
      RH(K)=0.0000D0
 4252 CONTINUE
C
C     GAUSS-SEIDEL ITERATION FOR OMEGA
C
      DO 755 K=1,1
      DO 727 I=1,TOTNOD
      SUM=0
      IP1=REG(I)
      IP2=REG(I+1)-1
      CNT=0
      DO 729 L=IP1,IP2
      IF(IRN(L).LT.I)GO TO 728
      IF(IRN(L).EQ.I)GO TO 729
      SUM=SUM+STIF(L)*GU(IRN(L))
      GO TO 729
  728 CNT=CNT+1
      SUM=SUM+STIF(L)*GU(IRN(L))
  729 CONTINUE
      IP3=IP1+CNT
      GU(I)=(RH(I)-SUM)/STIF(IP3)
  727 CONTINUE
  755 CONTINUE
 8011 FORMAT(D21.14)
C
C     ASSEMBLY OF LOAD VECTOR FOR RHS=OMEGA
C
      CALL VECNUL(RH,IRH,IRH,ITEST)
      DOFEL=NODEL*DOFNOD
      CALL QQUA4(WGHT, IWGHT, ABSS, IABSS,JABSS, NQP, ITEST)
      DO 6100 NELE=1,TOTELS
      CALL VECNUL(BEV,IBEV,DOFEL,ITEST)
      CALL DIRECT(NELE,ELTOP,IELTOP,JELTOP,NF,INF,JNF,
     *     DOFNOD,STEER,ISTEER,ITEST)
      DO 6091 IQUAD=1,NQP
```

```
      S(1)=0.00
      DO 6086 I=2,5
      S(I)=S(I-1)+GU(STEER(I-1))*YE(IQUAD,I-1)
 6086 CONTINUE
      DO 6085  K=1,DOFEL
      LN(K)=XE(NELE,IQUAD,K)*S(5)
 6085 CONTINUE
      CALL VECADD(BEV,IBEV,LN,ILN,DOFEL,ITEST)
 6091 CONTINUE
      CALL ASRHS(RH,IRH,BEV,IBEV,STEER,ISTEER,
     *NODEL,ITEST)
 6100 CONTINUE
C
C     APPLICATION OF BOUNDARY
C     CONDITIONS FOR PSI
C
      DO 6115 I=1,2
      DO 6120 J=1,COUNT
      K = BDCND(I,I,J+3)
      X = COORD(K,1)
      IF(COORD(K,2).EQ.0)GO TO 3
      TY=DSQRT((1-COORD(K,2)**2)/COORD(K,2)**2)
      Y=DATAN(TY)
      GO TO 2
    3 Y=PI/2.0
    2 GO TO(6125,6135),I
 6125 POD=REG(K)+DIAG(K)-1
      RH(K)=STIF(POD)*H1(X,Y)
      GO TO 6120
 6135 POD=REG(K)+DIAG(K)-1
      RH(K)=STIF(POD)*H3(X,Y)
 6120 CONTINUE
 6115 CONTINUE
      DO 6253 K=1,ICP,COUNT
      RH(K)=0.0000D0
 6253 CONTINUE
      DO  6252 K=COUNT,ICCP,COUNT
      RH(K)=0.0000D0
 6252 CONTINUE
C
C     GAUSS-SEIDEL ITERATION FOR PSI
C
      DO 766 K=1,1
      DO 827 I=1,TOTNOD
      SUM=0.0
      IP1=REG(I)
      IP2=REG(I+1)-1
      CNT=0
      DO 829 L=IP1,IP2
      IF(IRN(L).LT.I)GO TO 828
      IF(IRN(L).EQ.I)GO TO 829
      SUM=SUM+STIF(L)*ANEW(IRN(L))
      GO TO 829
  828 CNT=CNT+1
      SUM=SUM+STIF(L)*ANEW(IRN(L))
  829 CONTINUE
      IP3=IP1+CNT
```

```
      ANEW(I)=(RH(I)-SUM)/STIF(IP3)
  827 CONTINUE
  766 CONTINUE
C
      JTCC=JTC-IPOINT
      IF(JTCC.GT.0.001)GO TO 433
      JTC=JTC+20
      JTC4=MOD(JTC,40)
      IF(JTC4.LT.00001.0)GO TO 434
      IF(JTC4.GT.0.001)GO TO 435
  434 OPEN(FILE='GUANEW2',UNIT=8,STATUS='UNKNOWN')
      DO 756 I=1,TOTNOD
      WRITE(8,8011)GU(I)
  756 CONTINUE
      DO 747 I=1,TOTNOD
      WRITE(8,8011)ANEW(I)
  747 CONTINUE
      CLOSE(8)
      GO TO 433
  435 OPEN(FILE='GUANEW',UNIT=9,STATUS='UNKNOWN')
      DO 977 I=1,TOTNOD
      WRITE(9,8011)GU(I)
  977 CONTINUE
      DO 978 I=1,TOTNOD
      WRITE(9,8011)ANEW(I)
  978 CONTINUE
      CLOSE(9)
  433 OPEN(FILE='FJTC',UNIT=10,STATUS='UNKNOWN')
      WRITE(10,8010)JTC
      CLOSE(10)
      DO 777 I=1,TOTNOD
      RH(I)=ANEW(I)
  777 CONTINUE
      IF(IPOINT.GT.0.5)GO TO 7777
      WRITE(1,8041)
      WRITE(NOUT,8041)
C
C     CALCULATION OF THE INTEGRAL 'I'
C
 7777 DO 1017 I=1,COUNT
      IW(I)=(GU(INTRA(I))-GU(I))/DENOMA
      IW(I)=(IW(I)-2*GU(I))
 1017 CONTINUE
      IW(2)=4.0*IW(2)
      ICT1=COUNT-1
      ICT2=COUNT-2
      IW(3)=2.0*IW(3)
      DO 1018 I=4,ICT1,2
      IW(I)=IW(I)*4.0+IW(I-2)
 1018 CONTINUE
      DO 1019 I=5,ICT2,2
      IW(I)=IW(I)*2.0+IW(I-2)
 1019 CONTINUE
      TOTAL=IW(1)+IW(ICT2)+IW(ICT1)+IW(COUNT)
      DELTA=2.0/((COUNT-1)*1.0)
      AREA=DELTA*TOTAL/3.0
C
```

```
      RMARG=20.0000D0
      RM=DMOD(IPOINT,RMARG)
      SLOP=SLOPE+AREA/DIV+ALOG(2.0)
      ITER=IPOINT+1
      IPOINT=IPOINT+1
      ALPHA=0.0
      OPEN(FILE='FBETA',UNIT=11,STATUS='UNKNOWN')
      WRITE(11,8011)BETA
      CLOSE(11)
C
C     CALCULATION OF BETA
C
      DO 479 I=1,COUNT
      ITC3=COUNT3+I
      DGU(I)=(GU(ITC3)-GU(INTRB(I)))/DENOMB
      DANEW(I)=(ANEW(ITC3)-ANEW(INTRB(I)))/DENOMB
  479 CONTINUE
      BETA=LMULT1*((4/3.0)*DANEW(IDGU)-3-2*LOG(RR))
      BETA=BETA+LMULT2*(3-2*LOG(RR)+(2/3.0)*RR**2*DGU(IDGU))
C
      IF(ABS(RM).GT.0.005)GO TO 39
      BETAP=BETA+0.1972246
      WRITE(1,8039)SLOP,AREA,ITER
      WRITE(1,2057)
C
C     COMPARING THE NUMERICAL DERIVATIVES FOR
C     OMEGA AND PSI WITH THE CORRESPONDING
C     THEORETICAL VALUES
C
      WRITE(1,8039)DGU(IDGU),DANEW(IDGU)
      WRITE(1,8039)DRGU(IDGU),DRANEW(IDGU)
C
      WRITE(NOUT,8039)SLOP,AREA,ITER
   39 IF(IPOINT.LT.20000)GO TO 725
      STOP
 8050 FORMAT(//41H              MATCH RADIUS 020 MESH 140X20 //)
 8010 FORMAT(16I5)
 8023 FORMAT(//14H  INTO LOOP 23 )
 8037 FORMAT(14H  INTO LOOP 36 )
 8041 FORMAT(//42H               SLOPE                INTGR  //)
 8043 FORMAT(17H STIFFNESS MATRIX )
 8038 FORMAT(7H HBAND )
 8036 FORMAT(1H , 110F10.5)
 8039 FORMAT(2F20.5,I8)
 8020 FORMAT(I5, 6F12.5)
      END
      SUBROUTINE ASRHS(RHS, IRHS, VALUE, IVALUE, STEER, ISTEER,
     *     DOFEL, ITEST)
C-------------------------------------------------------------
      INTEGER DOFEL, ERRMES, IERROR, IRHS, ISTEER, ITEST, IVALUE,
     *     K, L, STEER
      DOUBLE PRECISION RHS, SRNAME, VALUE
      DIMENSION RHS(IRHS), STEER(ISTEER), VALUE(IVALUE)
      DATA SRNAME /8H ASRHS   /
                      IF (ITEST.EQ.-1) GO TO 1010
                      IERROR = 0
                      IF (ISTEER.LT.DOFEL) IERROR = 3
```

```
                          IF (IVALUE.LT.DOFEL) IERROR = 2
                          IF (DOFEL.LE.0) IERROR = 1
                          ITEST = ERRMES(ITEST,IERROR,SRNAME)
1010                      IF (ITEST.NE.0) RETURN
      DO 1030 K=1,DOFEL
      IF (STEER(K).EQ.0) GO TO 1030
      L = STEER(K)
                          IF (ITEST.EQ.-1) GO TO 1020
                          IERROR = 0
                          IF (L.GT.IRHS) IERROR = 4
                          ITEST = ERRMES(ITEST,IERROR,SRNAME)
                          IF (ITEST.NE.0) RETURN
1020  RHS(L) = RHS(L) + VALUE(K)
1030  CONTINUE
      RETURN
      END
```

PROGRAM 3

THIS PROGRAM CONTINUES THE RUN IN THE EVENT OF DOWN-TIME DURING THE RUNNING OF PROGRAM 2.

```
C
C
C                    THIS PRIOGRAM CONTINUES THE RUN
C                        IN THE EVENT OF DOWN-TIME
C                  DURING THE RUNNING OF THE PREVIOUS PROGRAM
C
C
      INTEGER BDCND,BTYPES,DIMEN,DOFEL,DIAG,
     *     DOFNOD,ELNUM,ELTOP,ELTYP,I,IABSS,
     *     ILOAD,ICNT,LL,IELTOP,INF,JELTOP,
     *     IQUAD,ISTEER,ITEST,IWGHT,REG,J,JABSS,IRN,POD,
     *     NF,NIN,NODEL,NODNUM,NOUT,NQP,JNF,K,NELE,
     *     STEER,TOTDOF,TOTELS,TOTNOD,CCC,COUNT1,COUNT2,
     *     COUNT3,COUNT,ILN,IBEV,IRH,BCD1,BCD2,
     *     ICP,ICCP,INTRA,INTRB
      DOUBLE PRECISION ABSS,DENOMA,SUM,STIF,SINARG,,MULT1,MULT2,
     *     COORD,DTPD,ELK,FUN,GDER,DENOMB,TY,BETAP,
     *     GDERT,GEOM,JAC,JACIN,LDER,P,PD,DIV,
     *     SCALE,WGHT,X,XX,Y,YY,LOAD,LO,GU,IPOINT,S(5),
     *     WGA,H1,H2,MID,F,R,MGDER(2,4),IW,SLOPE,ARGLOG,
     *     FMD,PI,LN,BEV,RH,OMVAA,SLOP,FN,RSTAR,GSTAR,RMARG,
     *     COEFF,XE(3000,4,4),YE(4,4)
      DIMENSION ABSS(2,9),
     *     DTPD(4,4),
     *     ELK(4,4),FUN(4),GDER(2,4),GDERT(4,2),GEOM(4,2),
     *     JAC(2,2),JACIN(2,2),LDER(2,4),P(2,2),PD(2,4),
     *     STEER(4),WGHT(9),MID(2,2),F(4,2),R(2),LN(4),
     *     FMD(4,4),BEV(4),LO(6561)
C
C               PROBELM SIZE DEPENDENT ARRAYS
C
      DIMENSION LOAD(6561),GU(6561),COORD(6561,2),
     *     ELTOP(6561,6),NF(6561,1),COEFF(3000,9),
     *     ANEW(6561),RH(6561),RSTAR(6561),GSTAR(6561),
     *     OMVAA(81),OMVAB(81),INTRA(81),INTRB(81),
     *     IW(81),BCD1(5,81),BCD2(5,81),
     *     BDCND(2,2,81),REG(6562),IRN(65620),DIAG(6561),
     *     STIF(65620)
      COMMON/A/ELTOP
      COMMON/B/DIAG
      COMMON/C/COORD
      COMMON/D/RH,ANEW,NF
      COMMON/E/ABSS,BCD1,BCD2,BDCND,BEV,DTPD,
     *     ELK,F,FMD,FUN,GDER,
     *     GDERT,GEOM,GU,INTRA,INTRB,IW
      COMMON/F/JAC,JACIN,LDER,LN,LOAD,MGDER,MID,
     *     OMVAA,OMVAB,P,PD,R,S,STEER,
     *     W1,W2,WGHT,REG,IRN
      COMMON/G/STIF,COEFF
      COMMON/H/GSTAR,RSTAR
      COMMON/I/XE
      COMMON/J/YE
C
      DATA IABSS/2/,ISTEER/4/,
     *     IWGHT/9/, JABSS/9/,JNF/1/,
     *     SCALE /1.0D+25/,ILOAD/6561/
C
```

```
C               PROBELM SIZE DEPENDENT DATA STATEMENTS
C
      DATA IELTOP/6561/,INF/6561/,JELTOP/6/,
     *     IRH/6561/,IBEV/4/,ILN/4/,
     *     RR/10.0/,BETA/-1.3/,COUNT/21/,
     *     MULT1/0.5/,MULT2/0.5/
C
      DATA NIN /5/, NOUT /6/
C
C     WGA(XX,YY) = 1.0*((DSIN(YY)**2)/34.0)*(90*XX**2-372/XX)
      WGA(XX,YY)=-0.5*(-2*BB/XX+10*DD*XX**2)*DSIN(YY)**2
      H1(XX,YY)=0.0000D0
C     H2(XX,YY)=-(DSIN(YY)**2*2.0)
      H2(XX,YY)= -(DSIN(YY)**2*0.5*RR**2)
C     H3(XX,YY)=-0.5*(BETA*RR+RR**2)*DSIN(YY)**2
      H3(XX,YY)=(3*(RR*BETA+2*RR*ALOG(RR))/4.0)*DSIN(YY)**2
C     H3(XX,YY)=-(DSIN(YY)**2*RR**2)/2.0
C     H3(XX,YY)=0.0000
C
C                              SET ITEST FOR FULL CHECKING
C
      ITEST = 0
      PI=4.0D0*DATAN(1.0D0)
C      AA=0.5
C      BB=-1.5
C      CC=1.0
C      DD=0.0
 9088 FORMAT(10F10.5)

C:                 READING DATAFILES CREATED BY
C:                     THE PREVIOUS PROGRAM.
      OPEN(FILE='ABCD',UNIT=12,STATUS='UNKNOWN')
      READ(12,9088)AA
      READ(12,9088)BB
      READ(12,9088)CC
      READ(12,9088)DD
      CLOSE(12)
      FN=24*RR*(1+RR+RR**2+RR**3+RR**4)/
     *          ((RR-1)**3*(4+7*RR+4*RR**2)*1.0)
      ARGLOG=2*AA+2*CC+12*DD
C
      DIV=4*(5*AA+2*CC)/3.0
      SLOPE=2*(0.75*(2*DLOG(ARGLOG)/3.0-5/9.0)
     *       +(4*AA+2*CC+4*DD)/DIV)
C
C     WRITE(1,8036)DIV,SLOPE,ARGLOG
C
C                      INPUT OF NODAL GEOMETRY
C
      OPEN(FILE='LLDATA',UNIT=7,STATUS='OLD')
      OPEN(FILE='FJTC',UNIT=10,STATUS='OLD')
      READ(10,8010)JTC
      OPEN(FILE='FBETA',UNIT=11,STATUS='OLD')
      READ(11,8011)BETA
      CLOSE(11)
      READ(7,8010)TOTNOD
      READ(7,8010)TOTELS
```

```
      ITOT1=TOTNOD+1
      DO 4551 I=1,ITOT1
      READ(7,8010)REG(I)
 4551 CONTINUE
      READ(7,8010)IPEND
      READ(7;8010)IPEND1
      READ(7,8010)IPENDT
      DO 4552 I=1,IPEND
      READ(7,8011)STIF(I)
 4552 CONTINUE
      DO 7559 I=1,IPEND
      READ(7,8010)IRN(I)
 7559 CONTINUE
      DO 4553 I=1,TOTNOD
      READ(7,8010)DIAG(I)
 4553 CONTINUE
      DO 4554 I=1,TOTNOD
      READ(7,8011)LO(I)
 4554 CONTINUE
      DO 754 I=1,4
      DO 753 J=1,4
      READ(7,8011)YE(I,J)
  753 CONTINUE
  754 CONTINUE
C
      DO 889 I=1,TOTELS
      DO 888 J=1,4
      DO 887 K=1,4
      READ(7,8011)XE(I,J,K)
  887 CONTINUE
  888 CONTINUE
  889 CONTINUE
      JTC4=MOD(JTC,40)
      IF(JTC4.GT.0.01)GO TO 678
      OPEN(FILE='GUANEW2',UNIT=9,STATUS='OLD')
      DO 4591 I=1,TOTNOD
      READ(9,8011)GU(I)
 4591 CONTINUE
      DO 4592 I=1,TOTNOD
      READ(9,8011)ANEW(I)
 4592 CONTINUE
      CLOSE(8)
      GO TO 611
  678 OPEN(FILE='GUANEW',UNIT=8,STATUS='OLD')
      DO 6543 I=1,TOTNOD
      READ(8,8011)GU(I)
 6543 CONTINUE
      DO 6542 I=1,TOTNOD
      READ(8,8011)ANEW(I)
 6542 CONTINUE
      CLOSE(9)
  611 CONTINUE
      CLOSE(7)
      READ(NIN,8010) TOTNOD,DIMEN
      DO 1010 I=1,TOTNOD
      READ(NIN,8020) NODNUM,(COORD(NODNUM,J),J=1,DIMEN)
 1010 CONTINUE
```

```
      READ (NIN,8010) ELTYP, TOTELS, NODEL
      DO 1020 I=1,TOTELS
      READ (NIN,8010) ELNUM, (ELTOP(ELNUM,J+2),J=1,NODEL)
      ELTOP(ELNUM,1) = ELTYP
      ELTOP(ELNUM,2) = NODEL
 1020 CONTINUE
      READ (NIN,8010) DOFNOD
      BTYPES=1
      DO 2324 I=1,COUNT
      J=I+3
      BCD1(1,J)=I
      BCD2(1,J)=I+TOTNOD-COUNT
 2324 CONTINUE
      DO 1075 J=1,COUNT
      JJ=J+3
      BDCND(1,1,JJ)=BCD1(1,JJ)
      BDCND(2,2,JJ)=BCD2(1,JJ)
 1075 CONTINUE
      CCC=COUNT+3
      K=COUNT+3
      DO 2071 I=1,COUNT
      INTRA(I)=BCD1(1,I+3)+COUNT
      INTRB(I)=BCD2(1,I+3)-COUNT
 2071 CONTINUE
      WRITE(1,8010) (INTRA(I),I=1,COUNT)
      WRITE(1,8010) (INTRB(I),I=1,COUNT)
      TOTDOF = 0
      DO 1050 I=1,TOTNOD
      DO 1040 J=1,DOFNOD
      TOTDOF = TOTDOF + 1
      NF(I,J) = TOTDOF
 1040 CONTINUE
 1050 CONTINUE
C
      ICP=TOTNOD-2*COUNT+1
      ICCP=TOTNOD-COUNT
C
      DO 1115 I=1,2
      DO 1120 J=1,COUNT
      K=BDCND(I,I,J+3)
      POD=REG(K)+DIAG(K)-1
      STIF(POD)=STIF(POD)*SCALE
 1120 CONTINUE
 1115 CONTINUE
C
      IPA=COUNT+1
      DO 1253 K=IPA,ICP,COUNT
      POD=REG(K)+DIAG(K)-1
      STIF(POD)=STIF(POD)*SCALE
 1253 CONTINUE
C
      IPB=2*COUNT
      DO 1252 K=IPB,ICCP,COUNT
      POD=REG(K)+DIAG(K)-1
      STIF(POD)=STIF(POD)*SCALE
 1252 CONTINUE
      COUNT1=COUNT+1
```

```
      COUNT2=2*COUNT+1
      COUNT3=TOTNOD-COUNT
      DENOMA=COORD(COUNT1,1)
      DENOMJ=DEXP(COORD(COUNT2,1))-DEXP(COORD(1,1))
      DENOMB=DEXP(COORD(TOTNOD,1))-DEXP(COORD(COUNT3,1))
      DO 1255 I=1,TOTNOD
      RH(I)=ANEW(I)
 1255 CONTINUE
      IPOINT=0.0000D0
      ICNT=0
      WRITE(1,8050)
      WRITE(NOUT,8050)
      WRITE(1,787)
  787 FORMAT(37H                     INTO ITERATIVE LOOP //)
      JTC=0
  725 DO 2073 I=1,COUNT
      JC1=I+1*COUNT
      OMVAA(I)=2*RH(JC1)/DENOMA/DENOMA
C     OMVAA(I)=-1.5*DSIN(SINARG)**2
C     SINARG=DACOS(COORD(INTRB(I),2))
C     OMVAB(I)=0.1*BETA*DSIN(SINARG)**2
      IF(COORD(INTRB(I),2).EQ.0)GO TO 47
      TY=DSQRT((1-COORD(INTRB(I),2)**2)/COORD(INTRB(I),2)**2)
      SINARG=DATAN(TY)
      GO TO 48
   47 SINARG=PI/2.0
   48 OMVAB(I)=(-3*(BETA-1+2*ALOG(RR))*DSIN(SINARG)**2)/(2.0*RR)
C  48 OMVAB(I)=2*RH(INTRB(I))/DENOMB/DENOMB
C    *            (RR-DENOMB)**2*DSIN(SINARG)**2/DENOMB
C    *         /DENOMB
 2073 CONTINUE
      CALL VECNUL(LOAD,ILOAD,ILOAD,ITEST)
      DO 2074 I=1,COUNT
      J=I+3
      LOAD(BCD1(1,J))=OMVAA(I)
      LOAD(BCD2(1,J))=OMVAB(I)
 2074 CONTINUE
      LL=TOTNOD-COUNT+1
      DO 1111 I=1,TOTNOD
      RH(I)=LC(I)
 1111 CONTINUE
      DO 4115 I=1,2
      DO 4120 J=1,COUNT
      K = BDCND(I,I,J+3)
      X = COORD(K,1)
      GO TO(4125,4135),I
 4125 POD=REG(K)+DIAG(K)-1
      RH(K)=STIF(POD)*LOAD(K)
      GO TO 4120
 4135 POD=REG(K)+DIAG(K)-1
      RH(K)=STIF(POD)*LOAD(K)
 4120 CONTINUE
 4115 CONTINUE
      ICCP=COUNT+COUNT*(COUNT-1)
      DO 4253 K=1,ICP,COUNT
      RH(K)=0.0000D0
 4253 CONTINUE
```

```
      DO  4252 K=COUNT,ICCP,COUNT
      RH(K)=0.0000D0
 4252 CONTINUE
      DO 755 K=1,1
      DO 727  I=1,TOTNOD
      SUM=0
      IP1=REG(I)
      IP2=REG(I+1)-1
      CNT=0
      DO 729 L=IP1,IP2
      IF(IRN(L).LT.I)GO TO 728
      IF(IRN(L).EQ.I)GO TO 729
      SUM=SUM+STIF(L)*GU(IRN(L))
      GO TO 729
  728 CNT=CNT+1
      SUM=SUM+STIF(L)*GU(IRN(L))
  729 CONTINUE
      IP3=IP1+CNT
      GU(I)=(RH(I)-SUM)/STIF(IP3)
  727 CONTINUE
  755 CONTINUE
 8011 FORMAT(D21.14)
C
C
      CALL VECNUL(RH,IRH,IRH,ITEST)
      DOFEL=NODEL*DOFNOD
      CALL QQUA4(WGHT, IWGHT, ABSS, IABSS,JABSS, NQP, ITEST)
      DO 6100 NELE=1,TOTELS
      CALL VECNUL(BEV,IBEV,DOFEL,ITEST)
      CALL DIRECT(NELE,ELTOP,IELTOP,JELTOP,NF,INF,JNF,
     *      DOFNOD,STEER,ISTEER,ITEST)
      DO 6091  IQUAD=1,NQP
      S(1)=0.00
      DO 6086  I=2,5
      S(I)=S(I-1)+GU(STEER(I-1))*YE(IQUAD,I-1)
 6086 CONTINUE
      DO 6085  K=1,DOFEL
      LN(K)=XE(NELE,IQUAD,K)*S(5)
 6085 CONTINUE
      CALL VECADD(BEV,IBEV,LN,ILN,DOFEL,ITEST)
 6091 CONTINUE
C
C :                 ASSEMBLY OF STIFFNESS MATRIX
C
      CALL ASRHS(RH,IRH,BEV,IBEV,STEER,ISTEER,
     *NODEL,ITEST)
 6100 CONTINUE
      DO 6115  I=1,2
      DO 6120  J=1,COUNT
      K = BDCND(I,I,J+3)
      X = COORD(K,1)
      IF(COORD(K,2).EQ.0)GO TO 3
      TY=DSQRT((1-COORD(K,2)**2)/COORD(K,2)**2)
      Y=DATAN(TY)
      GO TO 2
    3 Y=PI/2.0
    2 GO TO(6125,6135),I
```

```
6125 POD=REG(K)+DIAG(K)-1
     RH(K)=STIF(POD)*H1(X,Y)
     GO TO 6120
6135 POD=REG(K)+DIAG(K)-1
     RH(K)=STIF(POD)*H3(X,Y)
6120 CONTINUE
6115 CONTINUE
     DO 6253 K=1,ICP,COUNT
     RH(K)=0.0000D0
6253 CONTINUE
     DO  6252 K=COUNT,ICCP,COUNT
     RH(K)=0.0000D0
6252 CONTINUE
     DO 766 K=1,1
     DO 827 I=1,TOTNOD
     SUM=0.0
     IP1=REG(I)
     IP2=REG(I+1)-1
     CNT=0
     DO 829 L=IP1,IP2
     IF(IRN(L).LT.I)GO TO 828
     IF(IRN(L).EQ.I)GO TO 829
     SUM=SUM+STIF(L)*ANEW(IRN(L))
     GO TO 829
 828 CNT=CNT+1
     SUM=SUM+STIF(L)*ANEW(IRN(L))
 829 CONTINUE
     IP3=IP1+CNT
     ANEW(I)=(RH(I)-SUM)/STIF(IP3)
 827 CONTINUE
 766 CONTINUE
     JTCC=JTC-IPOINT
     IF(JTCC.GT.0.001)GO TO 433
     JTC=JTC+20
     JTC4=MOD(JTC,40)
     IF(JTC4.LT.0.001)GO TO 434
     IF(JTC4.GT.0.001)GO TO 435
 434 OPEN(FILE='GUANEW2',UNIT=9,STATUS='UNKNOWN')
     DO 756 I=1,TOTNOD
     WRITE(9,8011)GU(I)
 756 CONTINUE
     DO 747 I=1,TOTNOD
     WRITE(9,8011)ANEW(I)
 747 CONTINUE
     CLOSE(9)
      GO TO 433
 435 OPEN(FILE='GUANEW',UNIT=8,STATUS='UNKNOWN')
     DO 977 I=1,TOTNOD
     WRITE(8,8011)GU(I)
 977 CONTINUE
     DO 978 I=1,TOTNOD
     WRITE(8,8011)ANEW(I)
 978 CONTINUE
     CLOSE(8)
 433 OPEN(FILE='FJTC',UNIT=10,STATUS='UNKNOWN')
     WRITE(10,8010)JTC
     CLOSE(10)
```

```
C
C
C
      DO 777 I=1,TOTNOD
      RH(I)=ANEW(I)
  777 CONTINUE
      IF(IPOINT.GT.0.5)GO TO 7777
      WRITE(1,8041)
      WRITE(NOUT,8041)
 7777 DO 1017 I=1,COUNT
      IW(I)=(GU(INTRA(I))-GU(I))/DENOMA
      IW(I)=(IW(I)-2*GU(I))
 1017 CONTINUE
      IW(2)=4.0*IW(2)
      ICT1=COUNT-1
      ICT2=COUNT-2
      IW(3)=2.0*IW(3)
      DO 1018 I=4,ICT1,2
      IW(I)=IW(I)*4.0+IW(I-2)
 1018 CONTINUE
      DO 1019 I=5,ICT2,2
      IW(I)=IW(I)*2.0+IW(I-2)
 1019 CONTINUE
C
      TOTAL=IW(1)+IW(ICT2)+IW(ICT1)+IW(COUNT)
      DELTA=2.0/((COUNT-1)*1.0)
      AREA=DELTA*TOTAL/3.0
      RMARG=20.0000D0
      RM=DMOD(IPOINT,RMARG)
C     IF(ABS(RM).GT.0.01)GO TO 39
C
C
      SLOP=SLOPE+AREA/DIV+ALOG(2.0)
      ITER=IPOINT+1
      IPOINT=IPOINT+1
C
C     CALCULATION OF BETA
C
      DO 479 I=1,COUNT
      ITC3=COUNT3+I
      DGU(I)=(GU(ITC3)-GU(INTRB(I)))/DENOMB
      DANEW(I)=(ANEW(ITC3)-ANEW(INTRB(I)))/DENOMB
  479 CONTINUE
      BETA=LMULT1*((4/3.0)*DANEW(IDGU)-3-2*LOG(RR))
      BETA=BETA+LMULT2*(3-2*LOG(RR)+(2/3.0)*RR**2*DGU(IDGU))
C
C     BETA=AREA/6.0+2.0945349
      IF(ABS(RM).GT.0.005)GO TO 39
      BETAP=BETA+0.1972246
      WRITE(1,8039)SLOP,AREA,ITER
      WRITE(1,2057)
C
C     COMPARING THE NUMERICAL DERIVATIVES FOR
C     OMEGA AND PSI WITH THE CORRESPONDING
C     THEORETICAL VALUES
C
      WRITE(1,8039)DGU(IDGU),DANEW(IDGU)
```

```
      WRITE(1,8039)DRGU(IDGU),DRANEW(IDGU)
C
      WRITE(NOUT,8039)SLOP,AREA,ITER
   39 IF(IPOINT.LT.20000)GO TO 725
      STOP
 8050 FORMAT(//41H            MATCH RADIUS 020 MESH 140X20 //)
 8010 FORMAT(16I5)
 8023 FORMAT(//14H INTO LOOP 23 )
 8037 FORMAT(14H INTO LOOP 36 )
 8041 FORMAT(//42H                SLOPE                INTGR  //)
 8043 FORMAT(17H STIFFNESS MATRIX )
 8038 FORMAT(7H HBAND )
 8036 FORMAT(1H , 110F10.5)
 8039 FORMAT(2F20.5,I8)
 8020 FORMAT(I5, 6F12.5)
      END
      SUBROUTINE ASRHS(RHS, IRHS, VALUE, IVALUE, STEER, ISTEER,
     *     DOFEL, ITEST)
C-----------------------------------------------------------------
      INTEGER DOFEL, ERRMES, IERROR, IRHS, ISTEER, ITEST, IVALUE,
     *     K, L, STEER
      DOUBLE PRECISION RHS, SRNAME, VALUE
      DIMENSION RHS(IRHS), STEER(ISTEER), VALUE(IVALUE)
      DATA SRNAME /8H ASRHS  /
                          IF (ITEST.EQ.-1) GO TO 1010
                          IERROR = 0
                          IF (ISTEER.LT.DOFEL) IERROR = 3
                          IF (IVALUE.LT.DOFEL) IERROR = 2
                          IF (DOFEL.LE.0) IERROR = 1
                          ITEST = ERRMES(ITEST,IERROR,SRNAME)
 1010                     IF (ITEST.NE.0) RETURN
      DO 1030 K=1,DOFEL
      IF (STEER(K).EQ.0) GO TO 1030
      L = STEER(K)
                          IF (ITEST.EQ.-1) GO TO 1020
                          IERROR = 0
                          IF (L.GT.IRHS) IERROR = 4
                          ITEST = ERRMES(ITEST,IERROR,SRNAME)
                          IF (ITEST.NE.0) RETURN
 1020 RHS(L) = RHS(L) + VALUE(K)
 1030 CONTINUE
      RETURN
      END
```

PROGRAM 4

THIS PROGRAM SOLVES THE DIFFERENTIAL EQUATION FOR PULSATILE FLOW. THE NORMALISED PRESSURE GRADIENTS ARE CALCULATED. THE NORMALISED MEAN PRESSSURE GRADIENTS AND THE FUNDAMENTAL NORMALISED PRESSURE GRADIENT AMPLITUDES AND PHASE ANGLES ARE OBTAINED.

```
C        THIS PROGRAM SOLVES THE DIFFERENTIAL EQUATION
C        FOR PULSATILE FLOW. THE NORMALISED MEAN
C        PRESSURE GRADIENTS, THE FUNDAMENTAL PRESSURE
C        AMPLITUDE AND PHASE ARE OBTAINED.
C
C
      INTEGER*4  I,J,NC,COUNT,L1,L2,L3,L4,
     *     NR,NUT,J2,NCC,NR1,CTR,CTR2,NVAL,
     *     NUT1,NUTM1,IX,JX,KX,MAX,
     *     MEM1,MEM2,NUM,FLAG
      REAL*8 V(151,2),R(151),T(2),PI,QNUM,TOTAL,N,EPS,F,
     *     DELT,DELR,W(2),MULTAE,NN,NNN,DRNNN,DSGN3,DSGN4,
     *     INT(160),PSTAR(40010),GRAD,DRN,DSGN5,DSGN6,
     *     AMPL,GRR1,GR1,ARG,DIFFP(40010),DSGN7,DSGN8,
     *     MEAN,DSGN1,DSGN2,PHASE,
     *     MN(20),MF(20),MEPS(20),
     *     ALPHA,AJ,BIJ,CIJ,EPSS,EPSTAR,
     *     A1,A2,A3,A4,A5,B1,B2,B3,B4,B5
      COMMON /A/ INTO
      COMMON/B/V
      COMMON/C/PSTAR
      COMMON/E/T
      COMMON/D/ INT
      COMMON/E/DIFFP
C
      DATA MN(1)/0.3D0/,MEPS(1)/0.1D0/,MF(1)/0.5D0/,
     *     MN(2)/0.5D0/,MEPS(2)/0.5D0/,MF(2)/1.0D0/,
     *     MN(3)/0.7D0/,MEPS(3)/1.0D0/,MF(3)/2.0D0/,
     *     MN(4)/0.9D0/,MEPS(4)/5.0D0/,MF(4)/3.0D0/,
     *     MN(5)/1.0D0/,MEPS(5)/10.0D0/,MF(5)/5.0D0/,
     *                                  MF(6)/7.0D0/,
     *                                  MF(7)/10.0D0/,
     *                                  MF(8)/15.0D0/,
     *                                  MF(9)/20.0D0/,
     *                                  MF(10)/30.0D0/,
     *                                  MF(11)/50.0D0/,
     *                                  MF(12)/60.0D0/,
     *                                  MF(13)/70.0D0/,
     *                                  MF(14)/80.0D0/,
     *                                  MF(15)/100.0D0/,
     *     NC/3/,NUT/1000/,NR/20/,NVAL/1/,MAX/10000/
 2333 FORMAT(//7X,'NUMBER')
 3333 FORMAT(41H    OF CYCLES   STEP/CYCLE   RADIAL DIV. )
 4444 FORMAT(1I12,1I15,1I13)
 9999 FORMAT(30H       FOURIER COEFFICIENTS:A'S)
 7777 FORMAT(30H       FOURIER COEFFICIENTS:B'S)
 4445 FORMAT(5X,'NO. OF SIGN OSCILLATIONS IN PRESSURE VALUES',I13)
      WRITE(6,2333)
      WRITE(6,3333)
      WRITE(6,4444)NC,NUT,NR
 1111 FORMAT(1F13.3,1F14.3,1F14.3)
 2222 FORMAT(41H       VISCOSITY       EPSILON          F  )
C
C                    VARIABLE LIST
      NCC=NC+1
      NUT1=NUT+1
      NUTM1=NUT-1
```

```
      DELR=1.0/(DBLE(FLOAT(NR)))
      NR1=NR+1
      DELT=1/(DBLE(FLOAT(NUT)))
      NRM1=NR-1
      COUNT=0
      PI=4.0D0*DATAN(1.0D0)
C
C                    INITIAL CONDITIONS
      DO 1  I=1,NR
      V(I,1)=0.0D0
    1 CONTINUE
      V(NR1,1)=0.0D0
C                    RADIAL DIVISION
      R(1)=0.0D0
      DO 6  I=1,NR
      R(I+1)=R(I)+DELR
    6 CONTINUE
C
      T(1)=0.0D0
      DO 1812 IX=4,4
      N=MN(IX)
      ALPHA=((3.0D0*N+1.0D0)/N)**N
      NN=N-1
      NNN=N+1
      DRNNN=(1.0D0/DELR)**NNN
      DO 1944 JX=2,2
      EPS=MEPS(JX)
      DO 1957 KX=5,5
      F=MF(KX)
      MULTAE=2.0D0*ALPHA*EPS*PI*F/8.0D0
      WRITE(6,2222)
      WRITE(6,1111)N,EPS,F
      WRITE(6,9999)
      WRITE(6,7777)
      WRITE(6,4321)
      WRITE(6,2057)
 2057 FORMAT(5X,43H -------------------------------------------)
      WRITE(1,1111)N,EPS,F
C:IMPLEMENTING FINITE DIFFERENCE SCHEME
C=====================================
      CTR=0
    7 CTR2=0
      NUM=0
      NEM1=1
      FLAG=0
      DO 2 J=1,NUT
      T(2)=T(1)+DELT
      T(2)=2.0D0*PI*T(2)
      T(1)=2.0D0*PI*T(1)
      J2=J+1
      V(NR1,2)=0.0D0
      W(1)=V(2,1)-V(1,1)
      GRR1=(4.0D0*V(NR,1)-V(NRM1,1))/(2.0D0*DELR)
      DSGN1=DABS(GRR1)**N
      DSGN2=DABS(W(1))**N
      V(1,2)=MULTAE*(-1.0D0)*DSIN(T(1))+2.0D0*DSIGN(DSGN1,GRR1)
     *        +4.0D0*DSIGN(DSGN2,W(1))*(1.0D0/DELR)**NNN
```

```
      V(1,2)=V(1,2)*DELT*8.0D0/(F*ALPHA)+V(1,1)
      DO 3 I=2,NR
      K2=I+1
      AJ=MULTAE*(-1.0D0)*DSIN(T(1))+2.0D0*DSIGN(DSGN1,GRR1)
      DRN=DELR**N
      DRN=1.0D0/DRN
      DSGN3=DABS(V(K2,1)-V(I,1))**N
      DSGN4=V(K2,1)-V(I,1)
      BIJ=(R(I)+DELR/2.0D0)*DSIGN(DSGN3,DSGN4)
      DSGN5=V(I,1)-V(I-1,1)
      DSGN6=DABS(DSGN5)**N
      CIJ=(R(I)-DELR/2.0D0)*DSIGN(DSGN6,DSGN5)
      V(I,2)=AJ+(1.0D0/R(I))*DRNNN*(BIJ-CIJ)
      V(I,2)=V(I,2)*DELT*8.0D0/(F*ALPHA)+V(I,1)
    3 CONTINUE
C
      CTR=CTR+1
      IF(CTR.NE.NVAL)GO TO 17
      CTR2=CTR2+1
      CTR=0
C     CALCULATING THE NON-DIMENSIONALISED FLOW RATE
      DO 12 L1=1,NR1
      INT(L1)=V(L1,2)*R(L1)*2.0D0
   12 CONTINUE
      INT(2)=4.0D0*INT(2)
      INT(3)=2.0D0*INT(3)
      DO 13 L2=4,NR,2
      INT(L2)=INT(L2)*4.0D0+INT(L2-2)
   13 CONTINUE
      DO 14 L3=5,NRM1,2
      INT(L3)=INT(L3)*2.0D0+INT(L3-2)
   14 CONTINUE
      TOTAL=INT(1)+INT(NR)+INT(NRM1)+INT(NR1)
      QNUM=DELR*TOTAL/3.0D0
C     MODIFYING THE NON-DIMENSIONALISED VELOCITIES
C     USING THE NON-DIMENSIONALISED FLOW RATE.
      DO 15 L4=1,NR1
      EPSS=1.0D0+EPS*DCOS(T(2))
      EPSTAR=3.0D0*(EPSS-QNUM)
      V(L4,2)=V(L4,2)+EPSTAR*(1.0D0-R(L4))
   15 CONTINUE
   17 CONTINUE
C
C                CALCULATING THE NORMALISED
C                PRESSURE GRADIENTS.
      GRAD=(4.0D0*V(NR,2)-V(NRM1,2))/(2.0D0*DELR)
      DSGN7=DABS(GRAD)**N
      GRAD=DSIGN(DSGN7,GRAD)
      GR1=(4.0D0*V(NR,1)-V(NRM1,1))/(2.0D0*DELR)
      DSGN8=DABS(GR1)**N
      IF(J.GT.1)GO TO 77
      PSTAR(1)=DSIGN(DSGN7,GR1)
      PSTAR(1)=PSTAR(1)/ALPHA
      DIFFP(1)=0.0D0
   77 CONTINUE
      PSTAR(J2)=(F/16.0D0)*(-2.0D0*PI*EPS*DSIN(T(2)))
     *            +GRAD/ALPHA
```

```
      DIFFP(J2)=PSTAR(J2)-PSTAR(J2-1)
      T(1)=T(2)/(2.0D0*PI)
      DO 44 KK=1,NR1
      V(KK,1)=V(KK,2)
   44 CONTINUE
      MEM2=-1
      IF(DIFFP(J2).GT.0)MEM2=1
      IF(MEM2.EQ.MEM1)GO TO 231
      MEM1=MEM2
      FLAG=FLAG+1
      IF(FLAG.LE.1)GO TO 2
      NUM=NUM+1
C     WRITE(6,11111)NUM
      IF(NUM.GT.MAX)GO TO 8
  231 FLAG=0
    2 CONTINUE
      WRITE(1,11111)NUM
C             FOURIER ANALYSIS OF
C             NORMALISED PRESSURE GRADIENTS.
      CALL FANAL(PSTAR,DELT,PI,NUT,NUT1,NUTM1,
     *           MEAN,A1,A2,A3,A4,A5,B1,B2,B3,
     *           B4,B5,AMPL,ARG)
 1235 FORMAT(1F14.5,2F10.5,1I10)
 1234 FORMAT(//1F14.5,11F10.5)
 1236 FORMAT(1F14.5,11F10.5)
      WRITE(6,1234)A1,A2,A3,A4,A5
      WRITE(6,1236)B1,B2,B3,B4,B5
      PHASE=DATAN(-B1/A1)
      PHASE=PHASE*180/PI
      COUNT=COUNT+1
 4321 FORMAT(34H          MEAN       AMPL        COUNT)
      WRITE(6,1235)MEAN,AMPL,PHASE,COUNT
      WRITE(1,1235)PHASE
      WRITE(1,1235)MEAN,AMPL,COUNT
      WRITE(1,1235)COUNT,MEAN
      CONTINUE
C
      IF(COUNT.EQ.NC)GO TO 8
      T(1)=0.0D0
      GO TO 7
11111 FORMAT(12I10)
    8 CONTINUE
      COUNT=0
      T(1)=0.0D0
      DO 112 I=1,NR1
      V(I,1)=0.0D0
  112 CONTINUE
 1957 CONTINUE
 1944 CONTINUE
 1812 CONTINUE
      WRITE(6,2057)
      WRITE(6,4445)NUM
      WRITE(6,2057)
      STOP
      END
C -----------------------------------------------------------------
      SUBROUTINE FANAL(PSTAR,DELT,PI,NUT,
```

```
*     NUT1,NUTM1,MEAN,A1,A2,A3,A4,A5,B1,B2,B3,B4,
*     B5,AMPL,ARG)
      INTEGER NUT1,NUT,NUTM1
      DOUBLE PRECISION PI,MEAN,A1,A2,A3,A4,A5,
*     B1,B2,B3,B4,B5,AMPL,ARG,
*     PSTAR(40010),
*     X,DELT,
*     TOTALC,TOTALS,TOTC(5),TOTS(5),
*     TOTALO,CF(5),SF(5),CE(5),
*     SE(5),CO(5),SO(5),CL(5),SL(5)
C
      DO 1 I=1,5
      CF(I)=2.0D0*PSTAR(1)
      SF(I)=0.0D0
      CE(I)=2.0D0*PSTAR(2)*DCOS(DELT*2.0D0*I*PI)
      SE(I)=2.0D0*PSTAR(2)*DSIN(DELT*2.0D0*I*PI)
      CO(I)=2.0D0*PSTAR(3)*DCOS(DELT*4.0D0*I*PI)
      SO(I)=2.0D0*PSTAR(3)*DSIN(DELT*4.0D0*I*PI)
      CL(I)=2.0D0*PSTAR(NUT1)*DCOS(DELT*NUT*I*2.0D0*PI)
      SL(I)=2.0D0*PSTAR(NUT1)*DSIN(DELT*NUT*I*2.0D0*PI)
    1 CONTINUE
C     WRITE(1,1000)(CF(I),I=1,5)
C     WRITE(1,1000)(SF(I),I=1,5)
C     WRITE(1,1000)(CE(I),I=1,5)
C     WRITE(1,1000)(SE(I),I=1,5)
C     WRITE(1,1000)(CO(I),I=1,5)
C     WRITE(1,1000)(SO(I),I=1,5)
C     WRITE(1,1000)(CL(I),I=1,5)
C     WRITE(1,1000)(SL(I),I=1,5)
 1000 FORMAT(10F10.4)
      DO 90 I=4,NUT,2
      X=(I-1)*DELT*2.0D0*PI
      DO 2 J=1,5
      CE(J)=CE(J)+2.0D0*PSTAR(I)*DCOS(J*X)
      SE(J)=SE(J)+2.0D0*PSTAR(I)*DSIN(J*X)
    2 CONTINUE
C     WRITE(1,1000)(CE(K),K=1,5)
C     WRITE(1,1000)(SE(K),K=1,5)
   90 CONTINUE
      DO 91 I=5,NUTM1,2
      X=(I-1)*DELT*2.0D0*PI
      DO 3 J=1,5
      CO(J)=CO(J)+PSTAR(I)*2.0D0*COS(J*X)
      SO(J)=SO(J)+2.0D0*PSTAR(I)*DSIN(J*X)
    3 CONTINUE
C     WRITE(1,1000)(CO(K),K=1,5)
C     WRITE(1,1000)(SO(K),K=1,5)
   91 CONTINUE
      DO 4 I=1,5
      CE(I)=CE(I)*4.0D0
      SE(I)=SE(I)*4.0D0
      CO(I)=CO(I)*2.0D0
      SO(I)=SO(I)*2.0D0
    4 CONTINUE
      TOTALC=CF(1)+CE(1)+CO(1)+CL(1)
      TOTALS=SF(1)+SE(1)+SO(1)+SL(1)
      DO 6 I=2,5
```

```
      TOTC(I)=CF(I)+CE(I)+CO(I)+CL(I)
      TOTS(I)=SF(I)+SE(I)+SO(I)+SL(I)
    6 CONTINUE
      A1=TOTALC*DELT/3.0D0
      B1=TOTALS*DELT/3.0D0
      A2=TOTC(2)*DELT/3.0D0
      B2=TOTS(2)*DELT/3.0D0
      A3=TOTC(3)*DELT/3.0D0
      B3=TOTS(3)*DELT/3.0D0
      A4=TOTC(4)*DELT/3.0D0
      B4=TOTS(4)*DELT/3.0D0
      A5=TOTC(5)*DELT/3.0D0
      B5=TOTS(5)*DELT/3.0D0
      ARG=A1*A1+B1*B1
      AMPL=DSQRT(ARG)
      PSTAR(2)=4.0D0*PSTAR(2)
      PSTAR(3)=2.0D0*PSTAR(3)
      DO 53 N2=4,NUT,2
      PSTAR(N2)=PSTAR(N2)*4.0D0+PSTAR(N2-2)
   53 CONTINUE
      DO 54 N3=5,NUTM1,2
      PSTAR(N3)=PSTAR(N3)*2.0D0+PSTAR(N3-2)
   54 CONTINUE
      TOTALO=PSTAR(1)+PSTAR(NUT)+PSTAR(NUTM1)+PSTAR(NUT1)
      MEAN=DELT*TOTALO/3.0D0
      RETURN
      END
```

PROGRAM 5

THIS PROGRAM CALCULATES THE NORMALISED MEAN PRESSURE GRADIENT FROM THE POWER LAW MODEL PERTURBATION ANALYSIS.

```
10 REM  PROGRAM TO CALCULATE NORMALISED MEAN PRESSURE GRADIENT
20 REM  FROM THE POWER LAW MODEL PERURBATION ANALYSIS
30 :
40 DIM Y(100),Z(100)
50 N=.9
60 E=10
70 ND=100
80 F(1)=1:F(2)=2:F(3)=3:F(4)=4:F(5)=6:F(6)=8
90 P1=3.1415927£
100 P2=2*P1
110 FOR I=0 TO ND
120 U=P2*I/ND
130 Q=1+E*COS(U)
140 Y(I)=(ABS(Q))^N*SGN(Q)
150 Z(I)=-E*P2*P2*COS(U)*(ABS(Q))^(1-N)
160 NEXT I
170 GOSUB 340
180 Q0=RESULT
190 GOSUB 400
200 Q1=RESULT
210 K1=N*(3*N+1)*Q1/(512*(2*N+1)^2*(5*N+3))
220 FOR I=1 TO 6
230 Q=Q0+F(I)*F(I)*K1
240 PRINT Q;" ";
250 NEXT
260 PRINT
270 F(1)=10:F(2)=15:F(3)=20:F(4)=25:F(5)=30:F(6)=35
280 FOR I=1 TO 6
290 Q=Q0+F(I)*F(I)*K1
300 PRINT Q;" ";
310 NEXT I
320 PRINT
330 END
340 S1=0:S2=0
350 FOR I=1 TO ND-1 STEP 2:S1=S1+Y(I):NEXT
360 FOR I=2 TO ND-2 STEP 2:S2=S2+Y(I):NEXT
370 RESULT=(Y(0)+4*S1+2*S2+Y(ND))/(3*ND)
380 RETURN
390 :
400 S1=0:S2=0
410 FOR I=1 TO ND-1 STEP 2:S1=S1+Z(I):NEXT
420 FOR I=2 TO ND-2 STEP 2:S2=S2+Z(I):NEXT
430 RESULT=(Z(0)+4*S1+2*S2+Z(ND))/(3*ND)
440 RETURN
```

PROGRAM 6

THIS PROGRAM CALCULATES THE THEORETICAL NORMALISED MEAN PRESSURE GRADIENTS WHEN FLUID INERTIA IS IGNORED.

```
C
C
C           THIS PROGRAM CALCULATES THE THEORETICAL
C           NORMALISED MEAN PRESSURE GRADIENTS.
C           FLUID INERTIA HAS BEEN IGNORED.
C
C
      INTEGER NR,NRP1,NRM1
      DOUBLE PRECISION DELR,INT(200),TOTAL,PR(10),
     *                 X(200),ARG,N(10),EPS(10)
      DATA EPS(1)/.1/,EPS(2)/0.5/,EPS(3)/1.0/,
     *     EPS(4)/5.0/,EPS(5)/10.0/,EPS(6)/50.0/,EPS(7)/100.0/,
     *     N(1)/.3/,N(2)/.4/,N(3)/.5/,N(4)/.6/,
     *     N(5)/.7D0/,N(6)/.8D0/,N(7)/.9D0/,N(8)/1.0D0/,
     *     NR/100/,NOUT/6/
C
C
  600 FORMAT(4X,'EPS',5X,1F3.1,8X,1I1,9X,1I1,8X,1I2,8X,
     * 1I2,8X,1I3)
  800 FORMAT(5X,'N')
      WRITE(NOUT,600)EPS(2),(IDINT(EPS(I)),I=3,7)
      WRITE(NOUT,800)
      DO 5 J=1,8
      DO 6 K=1,7
C
      PI=4.0D0*DATAN(1.0D0)
      NRM1=NR-1
      NRP1=NR+1
      DELR=2*PI/100.0D0
      DO 1 I=1,NRP1
      X(I)=(I-1)*DELR
    1 CONTINUE
      DO 2 I=1,NRP1
      ARG=X(I)
      IF(DABS(1+EPS(K)*DSIN(ARG)).LT.0.000001)GO TO 7
      INT(I)=(1+EPS(K)*DSIN(ARG))*DABS(1+EPS(K)*DSIN(ARG))
     * **(N(J)-1)
      GO TO 2
    7 INT(I)=0.0
    2 CONTINUE
      INT(2)=4.0D0*INT(2)
      INT(3)=2.0D0*INT(3)
      DO 3 I=4,NR,2
      INT(I)=INT(I)*4.0D0+INT(I-2)
    3 CONTINUE
      DO 4 I=5,NRM1,2
      INT(I)=INT(I)*2.0D0+INT(I-2)
    4 CONTINUE
      TOTAL=INT(1)+INT(NR)+INT(NRM1)+INT(NRP1)
      PR(K)=DELR*TOTAL/3.0D0
      PR(K)=PR(K)/(2.0D0*PI)
    6 CONTINUE
      WRITE(NOUT,500)N(J),(PR(KK),KK=2,7)
  500 FORMAT(/1F7.1,7F10.4)
    5 CONTINUE
      STOP
```